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Mathematics. — “On the Surface of the Trisecants of a Twisted Curve which has Multiple Points”. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of March 29, 1924).

§ 1. In this paper we shall derive how the formula of CAYLEY:

$$(n-2) \{h - \frac{1}{2} n(n-1)\}$$

for the order of the surface of the trisecants of a twisted curve of the order n that has h apparent double points, must be altered if this curve has one or more multiple points.

After that we shall give an application of the result by determining the order of the surface that is formed by the vertices of the plane pencils which contain three chords of an arbitrary twisted curve.

§ 2. First we consider a twisted curve of the n^{th} order with h apparent double points that has one m -fold point M , and we determine the order of the surface of the trisecants of this curve in a way which is analogous to that in which ZEUTHEN, *Abzählende Methoden* [139], finds the number of quadrisecants of an arbitrary twisted curve.

The chords of the curve k^n , which cut a given straight line l , form a surface Ω of the order $h + \frac{1}{2}n(n-1)$. For through a point of l there pass h chords of k^n , so that l is an h -fold straight line of Ω , and in a plane through l there lie $\frac{1}{2}n(n-1)$ more generatrices of Ω , which are the joins of the n points which k^n has in common with this plane. We assume two points A_1 and A_2 outside k^n and to any point P_1 , in which the plane through A_1 and a generatrix b of Ω cuts the curve k^n outside b , we associate all the points P_2 , which the plane through A_2 and the same generatrix b has in common with k^n outside b .

The straight lines b corresponding in this way to a point P_1 of k^n , are the generatrices of Ω which cut the line A_1P_1 outside P_1 . Now k^n is an $(n-1)$ -fold curve of Ω as $n-1$ of the chords of this curve which pass through a point of k^n , and which form a cone of

From this there follows resp. by the aid of the equations (2) and (1):

$$k' = n-2$$

$$k = 2-n.$$

The application of the formula of CAYLEY-BRILL gives, that the correspondence (P_1, P_2) has

$$(n-2) \{4h+m(m-1)\}$$

coincidences.

If the planes (A_1, b) and (A_2, b) coincide, we have a coincidence (P_1, P_2) in each of the $n-2$ points of intersection outside b of the plane (A_1, A_2, b) with k^n . This happens for each of the $h + \frac{1}{2}n(n-1)$ generatrices b of Ω that cut the straight line A_1A_2 . We put aside these coincidences. We must therefore diminish the number found above by

$$(n-2) \{h + \frac{1}{2}n(n-1)\}$$

and there remain

$$(n-2) \{3h - \frac{1}{2}n(n-1) + m(m-1)\}$$

coincidences.

Through the m -fold point M of k^n there pass $n-m$ generatrices b_1 of Ω which cut k^n outside M ; they are the generatrices which meet l of the cone of the order $n-m$ which projects k^n out of M . If a point P on k^n approaches M along one of the m branches through M , $n-m$ of the $n-1$ generatrices of Ω through P approach to the lines b_1 , which are therefore m -fold generatrices of Ω . The remaining $m-1$ lines b through P are transformed into the $m-1$ lines b_1 through M , which lie in the planes touching the chosen branch of k^n and one of the other $m-1$ branches in M and cutting l . Besides the $n-m$ lines b_1 there pass accordingly through M $\frac{1}{2}m(m-1)$ more single generatrices b_2 of Ω , so that M is a $\frac{1}{2}m(2n-m-1)$ -fold point of Ω .

If P gets into M along a branch of k^n , on each of the other $m-1$ branches the two points of $(n-m)$ pairs of points (P_1, P_2) , corresponding to the $n-m$ lines b of Ω that are transformed into lines b_1 , get into M . In this way the lines b_1 give rise to $(n-m)m(m-1)$ coincidences (P_1, P_2) lying in M . Further any line b_2 gives a coincidence (P_1, P_2) in M on the $m-2$ branches of k^n which do not belong to the leaf of Ω containing b_2 . Again we put aside these coincidences and we must therefore also diminish the number of coincidences of the correspondence (P_1, P_2) by

$$(n-m)m(m-1) + \frac{1}{2}m(m-1)(m-2).$$

We also find the latter if we investigate by the aid of the rule

of ZEUTHEN ¹⁾ how many coincidences of the correspondence (P_1, P_2) lie in M . For if P_1 moves out of M along one of the branches of k^n , $m(n-m) + \frac{1}{2}m(m-1) - (n-1) = (n-m)(m-1) + \frac{1}{2}(m-1)(m-2)$ generatrices of Ω will cut the line P_1A_1 outside P_1 , and each gives therefore a point P_2 which originally coincided with P_1 , belonging to the same branch of k^n . Hence in all $(n-m)m(m-1) + \frac{1}{2}m(m-1)(m-2)$ coincidences (P_1, P_2) lie in M .

If we put these aside, there remain:

$$(n-2) \{3h - \frac{1}{2}n(n-1)\} + \frac{1}{2}m(m-1)(m-2)$$

coincidences of the correspondence (P_1, P_2) .

These lie apparently in those points of k^n through which there pass trisecants of this curve cutting l . The last found number is therefore equal to three times the number of trisecants of k^n cutting l . Hence:

The trisecants of a twisted curve of the order n with h apparent double points which has an m -fold point and is for the rest general, form a surface of the order:

$$(n-2) \{h - \frac{1}{6}n(n-1)\} + \frac{1}{6}m(m-1)(m-2).$$

If we consider therefore a twisted curve with a double point, the formula of CAYLEY gives the right result. In this case the cone of the order $n-2$ projecting the curve out of the double point, splits off from the surface of the trisecants and for such a curve h is one less and the result of the above mentioned formula $n-2$ less than for a twisted curve of the same genus without a double point.

If k^n has more than one multiple point, namely: v_3 triple points, v_4 quadruple points, etc. . . . v_m m -fold points, we find in the same way that the surface of the trisecants is of the order

$$(n-2) \{h - \frac{1}{6}n(n-1)\} + \sum_{i=3}^{i=m} \frac{1}{6} v_i i(i-1)(i-2).$$

For a curve which has only triple points we must therefore add the number of these points to the result of the formula of CAYLEY.

If we take e.g. a k^4 consisting of three straight lines l_1, l_2, l_3 passing through one point, and an arbitrary straight line l_4 , $h=3$ and we find three for the order of the surface of the trisecants. Indeed this surface consists of the three plane pencils which lie in the planes that may be passed through each pair of the lines l_1, l_2 and l_3 , and which have the points of intersection of l_4 with these planes for vertices. For a k^6 consisting besides of l_1, l_2 and l_3 of a similar

¹⁾ Abzählende Methoden, [116]2.

triple of lines l_4, l_5 and l_6 , $h = 9$ and we find for the order of the surface of the trisecants 18. In this case there lie indeed three plane pencils of trisecants in each of the six planes which may be passed through two intersecting lines l . If finally we have a k^3 consisting of three triples of straight lines through one point, $h = 9$ and the result of our formula is 108. In this case we have in each of the nine planes through two intersecting lines l six plane pencils and besides 27 scrolls of trisecants.

§ 3. The formula derived in § 2 does not hold good if in a multiple point M k^n has three branches touching at that point the same plane μ . For let us imagine a generatrix b of Ω cutting two of these branches in points lying at distances from M that are infinitely small of the first order. We can always choose l in such a way that such a generatrix of Ω exists. The distances from μ of the two points mentioned are infinitely small of the second order whereas the planes (A_1, b) and (A_2, b) form a finite angle. The straight line b forms accordingly with μ an angle that is infinitely small of the second order, the same as the angle of the lines along which μ is cut by the planes (A_1, b) and (A_2, b) . Consequently the latter two planes cut the tangent at M to the third branch in two points, the distance of which is infinitely small of the second order and their distances to M are infinitely small of the first order if l is chosen arbitrarily. The same holds also good for the two points P_1 and P_2 in which the third branch is cut resp. by the planes (A_1, b) and (A_2, b) . According to the rule of ZEUTHEN, already used in § 2 the coincidence counts twice instead of once on each of the three branches to which the line b ,¹⁾ belonging to the other branches, gives rise and the result of § 2 must therefore be diminished by one, as we find the order of the scroll of the trisecants of k^n by dividing the remaining number of coincidences by three.

Consequently the formula derived in § 2, must be diminished by one for each triple of branches passing through the same multiple point of k^n and touching there at the same plane.

From this there ensues that for a curve with only multiple points where all branches have the same tangent plane, the formula of CAYLEY may be applied. This is the case if the curve is the intersection of two surfaces if in each multiple point of this curve at least one of the surfaces has a single point.

Let us take for instance a k^3 composed of a twisted cubic k^3 and

¹⁾ The three straight lines b_2 corresponding to the three pairs of branches, coincide here in a triple generatrix of Ω .

two lines l_1 and l_2 passing through the same point M of k^3 . For this curve $h=5$. By the aid of the formula of § 2 we find six for the order of the surface of the trisecants. This surface consists of the two scrolls formed by the chords of k^3 which cut l_1 or l_2 , and of the two plane pencils in the plane (l_1, l_2) that have the points of intersection outside M of k^3 with this plane as vertices. If k^3 touches the plane (l_1, l_2) in M , according to the result of this § the order of the surface of the trisecants of k^5 must be five. In this case there is indeed only one plane pencil of trisecants.

§ 4. The formula derived in § 2 may be applied in order to determine the order of the surface consisting of the points through which there pass three chords in one plane of a curve k^n with h apparent double points.

In the communication: "*On the plane pencils Containing Three straight lines of a Given Algebraic Congruence of Rays*"¹⁾ the order of the surface formed by the vertices of the plane pencils containing three rays of a congruence $\Gamma(\alpha, \beta)$ of the rank r has been determined. This has been done by the aid of a representation of a special linear complex with axis α on the points of space, through which the surface Ω of the straight lines of Γ that cut α , is transformed into a twisted curve γ . The plane pencils with their vertices on α containing three straight lines of Γ , are represented on the trisecants of γ which cut a β -fold line of intersection of this curve.

If we take for Γ the congruence of the chords of k^n , the number of triple points of γ is the same as the order t of the surface of the trisecants of k^n . For in this case the trisecants of k^n cutting α are triple generatrices of Ω .

As a rule γ does not have one tangent plane at these triple points. For in that case there would exist a bilinear congruence with directrix α which in a trisecant d cutting α , has two coinciding straight lines in common with the three leaves through d of Ω . If this congruence were not special, the second directrix would touch the three leaves of Ω in a point of d . But then the three leaves of Ω would have the same tangent plane in each point of d , as two of these leaves already touch each other in the point of intersection of d and α and in one of the points of intersection of d with k^n . The same holds good if the bilinear congruence is special. For in that case the two points of d where two of the leaves of Ω through d have the same tangent plane, would coincide in the

¹⁾ These Proceedings vol. XXVI, p. 522.

point of intersection of d with a , and as their tangent planes also coincide in a point of intersection of d with k^n , these leaves would touch each other in each point of d . But then the straight line a would apparently touch in its point of intersection with d the quadratic surface which is defined by the three tangents to k^n in the points of intersection of d with this curve, hence the surface of the trisecants of k^n , which may be avoided by a suitable choice of a .

The bilinear congruence composed of the plane (a, d) and the sheaf (a, d) satisfies the above mentioned condition, but it is represented on a pair of planes which both pass through the triple point of k^n and consequently it does not give a tangent plane to k^n at this point.

We find accordingly the order of the surface of the trisecants of γ by adding t to the result of the formula of CAYLEY. Hence for a congruence Γ consisting of the chords of a twisted curve k^n the order of the surface of the points through which there pass three straight lines of Γ in one plane, is found by adding the order of the surface of the trisecants of k^n to the general result mentioned above.

Accordingly the locus of the vertices of the plane pencils containing three chords of a twisted curve, the bisecants of which form a congruence (α, β) of the rank r and the trisecants of which form a scroll of the order t , is a surface of the order

$$\frac{1}{6} (\alpha - 2) \{6r - (\alpha - 1)(3\beta - \alpha)\} + t.$$

To express this result in the order n and the number of apparent double points of the twisted curve k^n , we must substitute resp. h , $\frac{1}{2}n(n-1)$ and $(n-2)\{h - \frac{1}{6}n(n-1)\}$ for α, β and t . We find the rank r of the congruence of the chords of k^n in the following way. We choose a straight line l , draw the h bisecants of k^n through a point P_1 of l and pass planes through l and these h lines. In each of these planes there lie $\frac{1}{2}(n-2)(n-3)$ chords of k^n that do not cut on k^n the chord in this plane through P_1 . We associate to P_1 the points of intersection P_2 of these chords with l . In these same way we find the points P_1 corresponding to P_2 . The correspondence (P_1, P_2) has $h(n-2)(n-3)$ coincidences, which, however, coincide in pairs, for if P_1 moves out of a coincidence, two points P_2 become different from P_1 . Hence the rank r of the congruence of the chords of k^n is equal to $\frac{1}{2}h(n-2)(n-3)$.

We find:

The locus of the vertices of the plane pencils containing three

chords of a twisted curve of the order n with h apparent double points, is a surface of the order

$$\frac{1}{12} \{2h^3 + 3h^2(n^2 - 9n + 10) - h(3n^2 - 63n + 92) - 2n(n^2 - 1)\}.$$

This result cannot be applied to a conic as on each straight line there lies the vertex of a plane pencil containing an infinite number bisecants of the curve.

If we substitute successively for n and h resp.: 3 and 1, 4 and 2, 4 and 3, 5 and 4, 5 and 5, 5 and 6, we always find the result zero. Indeed in all these cases on an arbitrary straight line there does not lie a vertex of a plane pencil containing three bisecants of the curve, because the plane of this pencil would have six points in common with the curve. If we did not add the order of the surface of the trisecants, in the last four cases we should find a negative result, and we cannot conclude in any other way from the derivation why the formula does not hold good for these cases.

§ 5. We notice that the following more general theorem may be pronounced:

If we have a congruence (α, β) of the rank r with an infinite number of triple rays which form a scroll of the order t , the locus of the vertices of the plane pencils containing three lines of this congruence, is a surface of the order

$$\frac{1}{6}(\alpha - 2)\{6r - (\alpha - 1)(3\beta - \alpha)\} + t.$$

The following example gives a verification of this formula. We imagine three crossing straight lines a_1, a_2 , and a_3 , and two straight lines b_1 and b_2 , which cut a_1, a_2 , and a_3 . For the congruence C , consisting of the straight lines which cut two crossing individuals of the five given lines and which is therefore composed of four bilinear congruences, $\alpha = \beta = 4$. A plane pencil with its vertex outside the given lines which contains three lines of C , must contain one line which cuts b_1 and b_2 , and two lines which each meet a pair of the lines a . Suppose that the latter two lines of C both cut the line a_i . In this case the plane of the pencil passes through a_i but also through a line b , because besides a_i this plane contains another transversal of b_1 and b_2 , and therefore an infinite number of transversals of b_1 and b_2 . If we choose e.g. a point P in the plane through a_i and b_i , three straight lines of C through P belong to a plane pencil. For the transversals through P of a_i and one of the two other straight lines a lie in the plane (a_i, b) together with the transversal of b_1 and b_2 . Accordingly the locus of the vertices of

the plane pencils that have three straight lines in common with C , consists of *six* planes.

We find the rank r of the congruence C by remarking that the quadratic surface of the lines cutting an arbitrary line l , b_1 and b_2 , touches each of the three quadratic surfaces consisting of the lines which cut l and two of the lines a , in two points on l . Consequently there are six points on l through which there pass two lines of C lying in a plane through l . The scroll which has a_1, a_2 and a_3 as directrices, consists of triple generatrices of C ; hence $t = 2$. If we substitute resp. 4, 4, 6 and 2 for α, β, r and t in the formula indicated above, we find indeed the result *six*. Also in this case it appears that it is really necessary to add the order of the surface of the triple generatrices to the result of the general formula.

The five given straight lines form a special case of the twisted curve for which $n = 5$ and $h = 4$. The application of our formula to this curve gave the result zero whereas in the special case we find a surface of the sixth order. This is not in contradiction with the principle of the conservation of the number. For in this special case there lie on each straight line the vertices of six plane pencils which each contains an infinite number of bisecants of the degenerate quintic. They are the points of intersection of l with the six planes (a_i, b_k) . However we found a finite result because C consists of only a part of the bisecants of the degenerate quintic.

The following theorem may be proved which indicates the cases that the theorem of this § does not hold good:

If a congruence contains a scroll of triple rays for which the six plane pencils that have the foci for vertices and the corresponding focal planes for planes, always lie in a special bilinear congruence, the order of the said scroll must not be added to the result of the formula which holds good for an arbitrary congruence.

Physics. — “*On the Radiation and the Temperature of the External Photospheric Layers*”. By Dr. J. SPIJKERBOER. (Communicated by Prof. W. H. JULIUS).

(Communicated at the meeting of March 29, 1924).

There recently appeared an article ¹⁾ “On the Radiation and Temperature of the External Photospheric Layers” by RAGNAR LUNDBLAD. There some conclusions are arrived at with regard to molecular scattering, absorption, and radiation in the outer photospheric layers of the sun, and with regard to the temperature in these layers. The conclusions are decisive and of a far-reaching nature. An introductory sentence as “Starting from the observations on the distribution of the energy over the sun’s disk, the optical properties of the photospheric layers and the state of radiation within them are examined as closely as possible with a minimum of a priori assumptions” might, however, possibly lead one to impose too great confidence in the results, at least on a first perusal.

I am of opinion that very serious objections may be raised to the way of treatment of this problem by LUNDBLAD, and to the conclusions communicated by him. It is my intention to discuss the following points in this paper:

1. the differential equation from which LUNDBLAD starts, is that for a *plane* layer, and the same as that of SCHWARZSCHILD; 2. the limiting conditions are put so that it is a priori assumed that if solely molecular scattering causes the change of intensity over the sun’s disc, this distribution of intensity must be independent of the wave-length; hence LUNDBLAD’s conclusion that the influence of molecular scattering is very small, rests entirely on an arbitrary supposition; 3. no sufficient grounds are advanced to support this supposition.

As the further conclusions of LUNDBLAD are based on the supposition that molecular scattering alone cannot be the cause of the distribution of intensity over the sun’s disc, there is the same arbitrariness in these conclusions as in the supposition.

But these conclusions should also be rejected (as I will show in

¹⁾ R. LUNDBLAD, *Astrophysical Journal*, 58, 113, 1923.

a fourth paragraph) because, when the question is treated if the sun's atmosphere cannot consist of deeper layers which absorb and scatter, and an outer layer which only scatters, neither the limiting condition to which I referred under 2, may be combined with the supposition of a plane layer mentioned under 1, nor the solution of the integral equations can be considered sufficient. And that when considering absorption alone or absorption with scattering within definite limits LUNDBLAD is led to conclusions which should be entirely eliminated, will also be discussed in the fourth paragraph.

1. In his above cited article LUNDBLAD writes on page 115 "strictly speaking, the quantity ξ is a function of r , because the angle of incidence against the successive layers slowly varies as the beam traverses distances which cannot be neglected in comparison with the radius of the sun. But since we need not take such enormous distances into consideration, we are allowed to treat ξ as a constant".

If one does not want to venture a priori on a supposition on the *depth* of the layers in the sun, which either through molecular scattering, or through absorption, or through their own radiation also exert an influence on the emitted radiation, it is erroneous to treat ξ as a constant in the differential equation of the problem. By not considering ξ as a function of r , the problem which would have to be put without special suppositions for a *spherical shell*, is reduced to that for a *plane layer*.

Accordingly LUNDBLAD's equation is perfectly equivalent to the equations drawn up by SCHWARZSCHILD for a plane layer, and which have also been used by me.

By writing: $J_\lambda(r, \xi) = b(x, i)$; $\alpha = \kappa$; $\beta = \sigma$; $dr = -dx$; $\xi = \cos i$; $E = E$; $G = \frac{1}{2} A$, LUNDBLAD's equation (1) passes for $\xi > 0$ into the second of SCHWARZSCHILD's equations (3)¹⁾:

$$\cos i \frac{db}{dx} = (\kappa + \sigma) b - J,$$

whereas, for $\xi < 0$, by assuming: $J_\lambda(r, \xi) = a(x, i)$; $\alpha = \kappa$; $\beta = \sigma$; $dr = -dx$; $+\xi = -\cos i$; $E = E$; $G = \frac{1}{2} A$, the same equation (1) passes into the first of SCHWARZSCHILD's equations (3):

$$\cos i \frac{da}{dx} = -(\kappa + \sigma) a + J,$$

in which $J = \frac{\sigma A}{2} + \kappa E$, and closely related to the function H of LUNDBLAD.

¹⁾ K. SCHWARZSCHILD, Sitzungsberichte der Kön. Preuss. Akad. der Wiss., 47, 1183, 1914.

That LUNDBLAD's function G is equivalent to $1/2$ A of SCHWARZSCHILD is implied in the transformations introduced above.

For ¹⁾

$$\begin{aligned}
 G &= \frac{1}{2} \int_{-1}^{+1} J_{\lambda}(r, \xi) d\xi = \frac{1}{2} \int_{-1}^0 J_{\lambda}(r, \xi) d\xi + \frac{1}{2} \int_0^{+1} J_{\lambda}(r, \xi) d\xi = \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} a(x, i) \sin i di + \frac{1}{2} \int_0^{\frac{\pi}{2}} b(x, i) \sin i di = \frac{1}{2} A.
 \end{aligned}$$

2. LUNDBLAD's integral equations (7) and (8) (also LUNDBLAD has been obliged to split up the solution for the cases $\xi > 0$ and $\xi < 0$, and has accordingly practically introduced SCHWARZSCHILD's b -radiation and a -radiation later) are not entirely equivalent to SCHWARZSCHILD's integral equations (6) and (7).

This is owing to the fact that LUNDBLAD, before he writes down the solution, makes a *second supposition*, i. e. that the effective optical mass is infinitely great. Consequently the term that indicates in SCHWARZSCHILD what is still present in the emitted radiation of the same direction, of the intensity incident on the effective mass that is not thought infinitely great, after it has penetrated the atmosphere, is not found in LUNDBLAD ($\xi > 0$). If in SCHWARZSCHILD $H = \infty$ is taken, the integral equations are again perfectly equivalent, as appears pretty easily when the above-given transformations are taken into consideration, and in connection with the significance of the "optical mass" introduced by LUNDBLAD (cf. also SCHWARZSCHILD loc. cit. p. 1187),

This second supposition of LUNDBLAD ($\mu = \infty$; loc. cit. p. 117) already includes that if the particular case of a merely scattering, but not absorbing, nor itself radiating layer is considered (LUNDBLAD; loc. cit. p. 126), an intensity *must* be found that has the same ratios for all wave-lengths for the different places on the disc. For then the wave-length plays a part in m and as the optical mass has been put ∞ , the layer is *thicker* for the great wave-lengths than for the small wave-lengths.

The distribution of intensity over the disc very certainly *not* being the same for different wave-lengths according to the observations, the second supposition could not but lead to the conclusion that the significance of the molecular scattering must be eliminated.

SCHWARZSCHILD has made the observation (loc. cit. p. 1197) that

¹⁾ J. SPIJKERBOER, Arch. néerl., III A, V, 1, 1918.

already for $H = \infty$ (hence for an "optical mass" 8) the distribution of the emitted intensity over the different directions will be equal, except for a few percentages, to the distribution which would be found for the limiting case $H = \infty$.

3. It is clear that this second supposition has a far-reaching influence, particularly in connection with the first premise. For if in a consideration on the phenomena of radiation in the photospheric layers the theory of a plane layer, without corrections, is to be premised, it must be borne in mind that the thickness of the layer is small with respect to the radius of the sun (cf. my thesis for the doctorate); if after this a merely scattering atmosphere is to be treated, for which the optical mass is infinitely great, at any rate greater than 8, this implies that this optical mass must be thought crowded in this layer of relatively slight thickness; if besides the supposition of the infinitely great optical mass for *all* wave-lengths is to be premised, it must be assumed that even for the *least* strongly scattered kinds of radiation, hence for the infra-red, the mass can be so tremendous in that layer of comparatively limited dimension, while for the strongly scattered wave-lengths (violet) it is then, of course, a fortiori infinitely great.

As I think I have shown, on such premises it is utterly useless to examine whether the results of the observations of the distribution of light, as they have been made among others at Mount Wilson, are in concordance with the theoretical results. It is known beforehand that this *cannot* be the case.

4. When LUNDBLAD (loc. cit. p. 128) treats the case that the sun's atmosphere would consist of deeper layers, which absorb (and radiate as a black body) *and* scatter, and an external layer which only scatters, the limiting condition $\mu = \infty$ is certainly as questionable as regards the supposition that the layer is to be considered as plane as in the case considered in paragraph 2.

I may be allowed to refer here to the latter part of my thesis for the doctorate (p. 162—166)¹⁾, where I have come to the conclusion by different ways that layers of the sun lying *very deep* (roughly calculated lying down to $\frac{1}{10}$ of the radius of the sun below the photospheric limit) contribute to the total radiation for red and infra-red. And yet there is no question there of an optical mass $= \infty$; it is put no more than 8.

But it seems to me that the treatment of this special case by

¹⁾ J. SPIJKERBOER, Proefschrift, Utrecht, 1917; Arch. néerl., III A, V, 108—112, 1918.

LUNDBLAD must besides be considered as incompatible with the scheme of his solution of the integral equations.

For LUNDBLAD puts (loc. cit. p. 117) that his "emissivity-function" H is a polynomium of the N^{th} degree, $H(m) = \sum_{i=0}^N a_i m^i$, and says that it follows from the physical meaning of $H(m)$ that H is a continuous function of m with a limited number of maxima and minima. He is of opinion that an approximation will therefore probably prove very good even for a comparatively small number N . Though this is in general correct in my opinion, it should be borne in mind that in the special case considered here for *small* values of m (i. e. in the outer regions) the layer may only be scattering, whereas for *greater* values of m (i. e. in the deeper layers) scattering and absorption must co-operate.

This condition can only be thought to be satisfied when in the polynomium above mentioned the coefficients a of the first terms (of the lower powers of m) chiefly express the influence of the scattering and perhaps in a small degree the influence of the absorption, and if the coefficients of the higher powers of m indicate the influence of absorption and auto-radiation or perhaps are also partly determined by the influence of the scattering.

According to the equations of 9, 14, and 15 of LUNDBLAD (loc. cit. p. 118 and 119) there exists a simple relation between the coefficients a_i of the polynomium under consideration and the coefficients b_i of a second polynomium which indicates what the distribution of intensity is over the sun's disc in its dependence on the place on the disc. It seems impossible to me to come to another conclusion than that in the special case considered in this paragraph where the outer layer is considered to be merely scattering, *less* or *nothing* will be found about the coefficients a which determine the influence of the absorption and auto-radiation according as the optical mass of the outer merely scattering layers remains more or less far below SCHWARZSCHILD's critical value 8 mentioned in paragraph 2. And then also this part of LUNDBLAD's train of reasoning contains, therefore, an arbitrary supposition, viz. that the optical mass in the merely scattering layer must be comparatively *small* (much less than 8).

That from the observations on the distribution of intensity over the disc, as they have been made at Mount Wilson, only the four coefficients of the lower powers of m can be determined, renders the whole treatment for this special case hazardous even apart from this arbitrary supposition.

When LUNDBLAD has calculated, for different wave-lengths, the

values of the coefficients of the four powers of m in question, he derives from this the function H and the function G for the same wave-lengths and for different values of m . And from the values of these functions he then concludes to an upper limit for the coefficient of scattering, if the whole atmosphere were absorbing and scattering. As the numerical results for the absorption and scattering coefficients have been obtained only after the cases of scattering alone or scattering alone in the outer layers have been erroneously eliminated, while the calculations of the functions H and G are founded on the supposition of an infinite optical mass and a plane layer, no practical value can be assigned to these numerical values either.

Before long I hope once more to discuss the frequently still incorrect views of the nature of the scattering in extensive gas masses, also in connection with an article by ABBOT, which appeared already earlier and which treats, besides the scattering in the sun's atmosphere the sharp solar limb.

It is not superfluous to point out here that LUNDBLAD does not take the irregular refraction into account at all.

Bussum, February 1924.

Chemistry. — "*The Determination of the Size of Invisible Particles in Emulsoid Sols*". By Prof. H. R. KRUYT and H. J. C. TENDELOO.

(Communicated at the meeting of March 29, 1924).

1. In a number of communications from this laboratory the theory has been developed¹⁾ on the ground of investigations made on sols of agar¹⁾, rubber²⁾, starch³⁾, and gelatine⁴⁾, that emulsoid sols are systems in which the disperse phase consists of polymolecular particles⁵⁾; these systems should, accordingly, not be considered as genuine solutions of large molecules.

This theory rested on the observed fact that the charge of the particles in these systems is of a capillary-electrical nature, which could be established by investigations on the influence of electrolytes in very small concentration on the viscosity of the sols. The theory concerning the relation between the size of the particles and the viscosity on one side (EINSTEIN)⁷⁾ and that between the increase of the viscosity in consequence of the electrical charge of the particles on the other side (VON SMOLUCHOWSKI)⁸⁾ has shown us the way in this respect. According to the view advocated by us the emulsoid sol is a hydrated suspensoid sol, in which the stability of the sol is governed both by electrical charge and hydration.

The researches on the influence of tannins on the behaviour of different emulsoid sols⁹⁾ have given a new support to this theory.

¹⁾ H. R. KRUYT and H. G. DE JONG, *Zeitschr. f. physik. Chemie.* **100**, 250 (1922); cf. H. G. DE JONG, *Diss. Utrecht* 1921.

²⁾ H. R. KRUYT and W. A. N. EGGINK, *These Proc.* Vol. XXVI, p. 43. W. A. N. EGGINK, *Rec. trav. chim.* **42**, 317 (1923).

³⁾ H. G. BUNGENBERG DE JONG, *Rec. trav. chim.* **43**, 189 (1924).

⁴⁾ H. G. BUNGENBERG DE JONG, *Rec. trav. chim.* **43**, 35 (1924).

⁵⁾ Cf. H. R. KRUYT, *Koll. Zeitschr.* **31**, 338 (1922).

⁶⁾ An investigation on the casein sol has been performed by Mr. H. LIER and will shortly be published. The results are in perfect harmony with the theory.

⁷⁾ A. EINSTEIN, *Ann. der Physik* **19**, 289 (1906); **34**, 591 (1911).

⁸⁾ M. VON SMOLUCHOWSKI, *Koll. Zeitschr.* **18**, 190 (1916).

⁹⁾ H. G. BUNGENBERG DE JONG, *Rec. trav. chim.* **42**, 437 (1923); **43**, 189 (1924).

It remained, however, desirable to ascertain how large the particles in an emulsoid sol really are. A direct determination is excluded, the sol being amicroscopical; in the ultra microscope only a TYNDALL cone is seen. A differentiated image, indeed, arises when a dehydratant is added to the sol, and it seemed at first as if in this way exclusively an increase of contrast was brought about, and therefore a counting of the particles was rendered possible; continued investigations have, however, thrown doubt upon the question if, and in how far, it is possible to bring about dehydration without any agglutination of particles taking place. Therefore, it remained desirable to find a method in which the size of the particles in the emulsoid sol is determined under circumstances which as little as possible give rise to changes in the sol.

We think we have succeeded in this and in what follows we give the result of preliminary measurements, made on the starch sol, which proved to be exceptionally suitable for the purpose.

§ 2. The relation between the viscosity of a sol η_s and the viscosity of the dispersant η_0 is given, in the case that particles are uncharged, by the following formula of EINSTEIN:

$$\eta_s = \eta_0 \left(1 + \frac{5}{2} \varphi \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which φ is the volume of the particles as fraction of the total volume. VON SMOLUCHOWSKI has extended this formula to the case that the particles are charged, through the following relation:

$$\frac{\eta_s - \eta_0}{\eta_0} = \frac{5}{2} \varphi \left[1 + \frac{w}{\eta r^2} \left(\frac{\zeta D}{2\pi} \right)^2 \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which w represents the specific resistance of the sol, D the constant of dielectricity, r the radius of the particle, which we wish to know, and ζ the potential difference in the double layer of the particle. This relation only contains the specific resistance, the potential difference of the double layer, φ the volume of the joint particles, and the viscosities which can be determined, as was discussed in earlier papers. Measurements of the electrical conductivity κ in the usual way according to KOHLRAUSCH yield the value w ; cataphoretic determinations made on the sol, yield the value of the potential difference according to the equation:

$$\zeta = \frac{4\pi l}{ED} \eta u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which E represents the potential difference applied at the elec-

trodes, l the distance of the electrodes and u the cataphoretic velocity of the particles.

Hence measurements of the cataphoretic velocity supply us with the value of the boundary plane potential ζ . These measurements are, however, not easy to carry out; we have applied such modifications to the existing apparatus that the measurements can be made with an accuracy of about 2%. Before long we will discuss this more at length.

As unknowns there are now left φ and r . When now in an emulsoid sol a number of determinations are made with different electrolyte concentrations, a relation according to formula (2) is obtained each time; hence r can always be calculated from two determinations by eliminating φ between the two equations.

When, therefore, a series of determinations are made on a sol, either with the same electrolyte, or with different electrolytes, and the viscosity, the specific conductivity, and the cataphoretic velocity is every time measured, values for r can be calculated by combining the results in couples.

In this the supposition is, however, made that φ and r themselves do not vary with the charge of the sol, which is continually modified by the addition of electrolyte.

Earlier investigations have already given support to the supposition that this is really the case in sols of agar¹⁾ and starch²⁾, which is decidedly not a priori certain for a sol of gelatine³⁾. The following results must point out, whether the different combinations yield constant values both for r and for φ or not.

§ 3. The subjoined table 1 contains the values for the viscosity, the specific conductivity, and the cataphoretic velocity for a number of $\frac{1}{2}$ perc. sols of starch, to which several electrolytes are added in the quantities given there. In the sixth column is found recorded the values of ζ calculated from the measurements of the cataphoretic velocities.

Table 2 gives the values for r and for φ , calculated by combining the results from the preceding table in couples.

4. It appears from the second table that constant values for φ and r actually appear from the different combinations given in the table, a result which confirms the supposition made.

¹⁾ Cf. footnote 1) on p. 377.

²⁾ Cf. footnote 2) on p. 377.

³⁾ Cf. footnote 4) on p. 377.

TABLE 1.

Number.	Added electrolyte.	η_s	$\kappa \times 10^{-6}$	$u \times 10^5 \frac{\text{cm.}}{\text{sec.}}$	$\zeta \times 10^5$
		abs. units	e.s.u.	per $\frac{\text{volt}}{\text{cm.}}$	e.s.u.
1	0.5 m.aeq. KCl	0.01036	78.29	3.70	1.56
2	1.0 m.aeq. KCl	0.01023	146.07	3.02	1.27
3	0.5 m.aeq. MgSO_4	0.01033	66.83	3.85	1.62
4	0.8 m.aeq. MgSO_4	0.01028	99.63	3.23	1.33
5	1.0 m.aeq. MgSO_4	0.01022	111.53	2.65	1.11
6	0.1 m.aeq. $\text{Co(NH)}_6\text{Cl}_3$	0.01016	24.54	2.94	0.98
7	0.3 m.aeq. $\text{Co(NH)}_6\text{Cl}_3$	0.01005	51.77	1.92	0.81

TABLE 2.

Number of the observation used	$\varphi \times 10^2$	r in $\mu\mu$
1 and 2	5.7	7.7
1 and 5	5.8	7.3
3 and 5	5.6	5.4
1 and 3	5.6	6.0
6 and 7	4.8	6.7
3 and 4	4.9	9.7

Of more importance, however, is the result obtained about the actual value of r , which may be put at about 7 $\mu\mu$.

The hydrated particles in the emulsoid sol of starch have, accordingly, a diameter of about 14 $\mu\mu$. Undoubtedly the hydrated particles of this dimension must be invisible in the ultra-microscope, and only give rise to a diffuse Tyndall-cone. It would be, however, absurd to consider such a gigantic particle as a chemical molecule, though kinetically (but then entirely formally) it may, of course, be considered as a *molecule*. Our result quite corroborates the view on which our former theories rested. For also the particle, when the hydration water is left out of account, is too big to be considered as a molecule.

According to table 2 the total volume of the hydrated particles

φ is about 5.5 % of the volume of the sol, which contains $\frac{1}{2}$ % of starch; if for the latter 1.3 is taken as Sp. Gr., the volume of the dry amyl is 0.4 %. Hence the volumes of the dry and the hydrated particle are to each other as 1 : 14. The radii are, therefore, to each other as 3 : 7. When, therefore, the water of hydration is not taken into account, a particle would remain of a radius of 3μ , hence of a diameter of 6μ . For this too the same considerations are valid, as were given just now for the hydrated particle.

The research is being continued with some other sols.

Utrecht, March 1924.

VAN 'T HOFF-Laboratory.

Physiology. — "*On the Action of Insulin*". By Dr. J. G. DUSSEER DE BARENNE and Dr. G. C. E. BURGER. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of March 29, 1924).

With regard to various fundamental questions about the action of insulin, opinions still differ widely, e.g. on the question whether the respiratory gaseous exchange, notably the respiratory quotient (R. Q.), is influenced by the substance. BANTING, BEST, MACLEOD and their co-workers, and also KROGH assert that insulin induces a rise of the R. Q. Others on the contrary, e.g. DUDLEY, LAIDLAW, TREVAN and BOOCK maintain that insulin evolves a lowering of the R. Q.

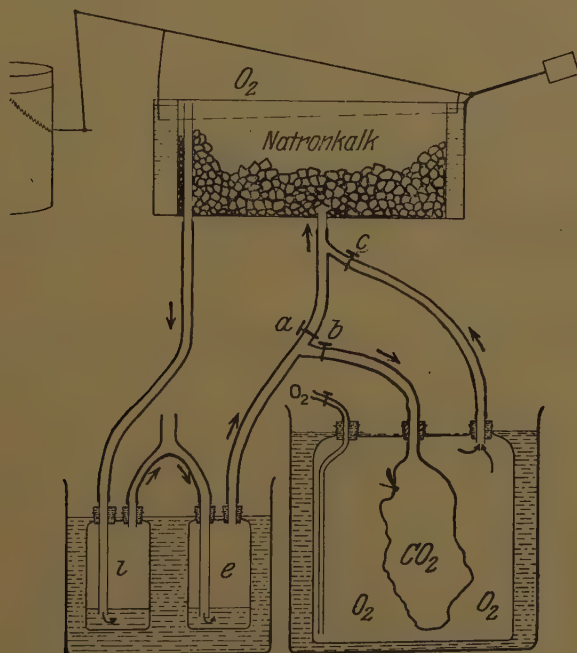
The latest number of the *Klinische Wochenschrift* (N°. 15, March 25, 1924, p. 555) contains a report of a communication by A. BORNSTEIN in the "Ärztlicher Verein" Hamburg on 15 Jan. last, in which he gives, according to the reporter WOHLWILL the following summary of the researches hitherto made into this subject and of the results of his own experiments: "Die Untersuchungen welche eine Steigerung des R. Q. by Insulinwirkung ergeben haben, woraus dann auf vermehrten Zuckerverbrauch geschlossen wurde, sind deshalb nicht zu verwerten, weil dabei nicht der Einfluss erhöhter Lungenventilation auf den R. Q. berücksichtigt ist. Zieht man nur solche Tierexperimente in Betracht, bei denen die Lungenventilation nachweislich auf gleicher Höhe gehalten wurde, so zeigt sich, dass Insulin keine oder nur eine ganz unbedeutende, innerhalb der Bestimmungsmethode gelegene Erhöhung des R. Q. bedingt".

We should not like to endorse the first statement expressed in this summary; in our opinion it cannot relate to the inquiries prosecuted in the laboratory of TORONTO nor to KROGH's researches upon the gaseous exchange under insulin. When we consider the fact that KROGH's experiments have been conducted on curarised rabbits under artificial respiration, the argument contained in the first sentence falls to the ground. Neither would we be responsible for the statement contained in the second sentence, which sums up BORNSTEIN's own results in this matter.

By means of our method of the "CO₂-trap" which enables us to record graphically the oxygen-consumption, the carbon-dioxid-output,

therewith the R. Q., and also the pulmonary ventilation, we have of late also studied the respiratory exchange under the influence of insulin and have attained results entirely opposite to those achieved by BORNSTEIN. We might, therefore, briefly record them here; surely the question of the respiratory exchange under insulin must be looked upon as one of the fundamental questions of the insulin-problem.

First of all we might set forth the principle of our new method, which may be done best in referring to Fig. 1. For a fuller discussion of this method we refer to our paper in the *Klinische Wochenschrift*, 1924, p. 395, and especially to that in the *Journal of Physiology*, Vol. 59, 1924.



Natronkalk = Soda lime.

Fig. 1

The oxygen-consumption is determined graphically with the aid of a small spirometer after KROGH¹⁾ (of ca 1,5 liters capacity for experiments on cats), to which is attached the "trap" for determining the CO₂-output:

The inspiratory-, and expiratory valves (watervalves after MÜLLER *i* and *e* of 2 mm. resistance only) are connected with the animal through the trachea-cannula and through rubber tubes with the spirometer, filled with oxygen. On the bottom there is sodalime for the absorption of the expired CO₂.

¹⁾ A. KROGH. Sur un appareil respiratoire enregistreur servant à déterminer l'absorption d'oxygène et les échanges caloriques chez l'Homme. C. R. de la Soc. de Biol. de Paris, 1922, Tome II, blz. 458.

When writing an O_2 -curve the tube is open at a , but the tubes, leading to and from the CO_2 -trap, are closed at b and c . The CO_2 given off by the animal during this O_2 -period, is absorbed by the soda-lime; only the unconsumed O_2 comes back again in the spirometer. The respiratory curve of the animal on the kymograph is a rhythmically moving, gradually falling curve; its fall at the end of an experimental period of 10—20 minutes, after calibration of the spirometer, gives the amount of O_2 , taken in by the animal.

For registration of a CO_2 -curve the tube is closed at a and the other tubes are opened at b and c . The animal now expires into a rubber balloon enclosed airtight in a large reservoir, filled with O_2 . This rubber balloon of 14 liters capacity in our experiments on cats, serves as the " CO_2 -trap", in which the expired air is caught as in a trap, so that it cannot mix with the O_2 in the spirometer, but can act volumetrically as if entering the spirometer. On reaching the trap the expired air extends this balloon, and expels exactly its volume of O_2 from the reservoir, in which the rubber balloon is suspended, back into the spirometer. The curve that is now traced by the respiration of the animal, is less abrupt in its descent than the O_2 -curve, because the expired CO_2 is not absorbed by the soda lime, but only retained in the trap and acts volumetrically as if it entered the spirometer. *The difference in fall between the CO_2 - and the O_2 -curve corresponds to the amount of expired CO_2 .* Of course due regard should be given to an even temperature of the whole apparatus; therefore the O_2 -reservoir and the trap it contains should be put in a water-bath or another heat-isolating device. (See Fig. 1).

The advantages of this method are manifest. The apparatus enables us to determine graphically the CO_2 -output, just as easily as KROGH's, or BENEDICT's method does for the O_2 -intake. As the bloodpressure may be studied from a bloodpressure-curve, so the bearing of all sorts of operations and substances upon the gaseous exchange may be represented graphically, while it can also be readily ascertained whether with contingent changes, as with an increase of the R. Q., hyperventilation comes into play. From these CO_2 -curves, in combination with the O_2 -curves, the R. Q. can be computed. But even at first glance some information about the R. Q. may be gathered from the CO_2 -curve. If this curve descends it implies a R. Q. smaller than 1, if it runs horizontally we have in this experimental period a R. Q. = 1, and if the curve rises above the horizontal, it means that the R. Q. during that rise is higher than 1.

Numerous problems may, therefore, be studied in this way; some of them we have already taken in hand, e.g. the problem of the gaseous exchange under the influence of insulin. Thus far our experiments were made on cats which had fasted for at least 24 hours, mostly 36—48 hours. The experiment proper was made on the decerebrate, normally breathing animal, when it had been left alone after the decerebration for one hour, at the very least, often much longer (3—4 hours), in order to allow the anesthetic (ether)

to disappear from the animal. In order to maintain a constant body-temperature, the animal was placed after the decerebration in a waterbath at 37°; only the paws and a small part of the chest and the abdomen being submerged, so that the breathing of the animal was not hindered by the pressure of the water.

After the anesthetic could be expected to have left the animal, first a "normal curve" of the O_2 -intake and of the CO_2 -output was registered, then insulin was administered during the tracing of a CO_2 -curve, and while more CO_2 -curves were being written, the course of the CO_2 -output was followed; finally the experiment was ended by an O_2 -curve. In the intervals between the several CO_2 -curves the balloon was emptied everytime and some c.c. of arterial blood were drawn for bloodsugar estimation (method of SHAFFER-HARTMANN, c.q. of COHEN TERVAERT ¹).

The result of the six experiments made up to now, is that in five of them a distinct rise of the R. Q. was observed after the injection of insulin, without the occurrence of hyperventilation; in two of them the R. Q. rose even above 1. In one animal the R. Q. was already 1 before the injection of insulin and maintained itself after the injection.

To instance this we subjoin the following protocol and fig. 2, representing the curves obtained.

Experiment of 23 Feb. 1924:

Castrated male cat, body-weight 3.10 kg. Ether anaesthesia, during which moderate excitation. Tracheal canula. Vagi left intact. Carotids ligated. (Barometer at 2 p.m. 767,5 mm.; at 8 p.m. 768,5 mm. Hg).

8³⁰ a. m. Blood I drawn (from carotid with potassium-oxalate).

8⁴⁵ a. m. Decerebration. Hypophysis not removed²). Animal in waterbath at 37° C.

11¹⁵ a. m. Animal quite well. Maximal decerebrate rigidity in the four legs.

11²⁵ a. m. Blood II drawn.

11⁴⁰ a. m. Hypophysis removed as a whole, sella turcica not cleaned²).

Curve 1. O_2 -curve.

12¹⁹—12³⁰ p. m. Maximal decerebrate rigidity in the four legs; temperature spirometer 11,9°—12°, rectal temperature of the animal 39,1° C.

12³³ p. m. Blood III drawn.

Curve 2. CO_2 -curve.

1⁵⁶—2⁰⁸ p. m. Maximal decerebrate rigidity in the four legs. Temperature spirometer 12°—12°. Rectal temperature 39°.

2¹³ p. m. Blood IV drawn.

¹) See D. G. COHEN TERVAERT. Nieuwe methoden voor bloedsuikerbepaling, Nederlandsch Tijdschrift voor Geneeskunde, 1921, II, p. 857.

²) These data are not important in our case. They are so, however, for another question that will be left for further discussion in a future paper.

Curve 3. CO₂—curve 3^a.

3⁰¹—3¹² p. m. Temperature spirometer 12,4⁰—12,8⁰. Rectal temperature 39⁰.

3⁰⁷ p. m. 4 units of Iletin Lilly (American insulin U 20) injected intravenously

at the point marked ↓; at *m* beginning of the manipulations.

Curve 3^b.

3²¹—3³⁴ p. m. Temperature spirometer 12,9⁰—12,9⁰. Rectal temperature 38,9⁰.

Curve 3^c.

3⁵⁷—4¹⁰ p. m. Maximal decerebrate rigidity in the four legs. On either side vivid reflexes of the pinnae, the corneae and the whiskers. Animal well. Temperature spirometer 12,9⁰—12,9⁰. Rectal temperature 38,8⁰.

4³⁶ p. m. Blood V drawn.

Curve 3^d.

4⁴¹—4⁵⁴ p. m. Temperature spirometer 12,9⁰—12,7⁰. Rectal temperature 38,8⁰.

Curve 4. O₂—curve.

5¹²—5²⁴. Decerebrate rigidity in right foreleg almost quite disappeared, still rather marked in left foreleg and in hindlegs. Cornea-reflex on either side +, reflexes of the pinnae and the whiskers —. Temperature spirometer 12,8⁰. Rectal temper. 39⁰.

7²² p. m. Blood VI drawn. Calibration of the spirometer: 200 cc. = 29 mm.

The rest of the protocol is not essential in this case, so it is omitted here.

Result of the bloodsugar determinations:

Blood I	.	.	.	.	.	.	1,9 ‰
Blood II	.	.	.	.	.	.	2,5 ‰
Blood III	{	before	Insulin	.	.	.	2,77 ‰
Blood IV		.	.	.	.	.	4,00 ‰
Blood V	{	after	Insulin	.	.	.	2,15 ‰
Blood VI		.	.	.	.	.	1,3 ‰

Result of the determinations of the gaseous exchange:

Curve 1 fall in 10 min. = 32,7 mm.; consequently O₂-consumption per kg.-hour

(reduced to 0⁰ and 760 mm. Hg. dry air) $\frac{200 \times 6 \times 32,7}{29 \times 3,10} \times 0,953 = 415,9$ cc.

Curve 2 fall in 10 minutes = 7,7 mm, consequently CO₂-output = 318,0 cc.

Curve 3^a fall in 10 minutes = 6,7 mm., consequently CO₂-output = 330,1 cc.; R.Q. = 0,79.

Curve 3^b fall in 9 min. = 3,7 mm., so in 10 min. = 4,1 mm.; consequently the CO₂-output = 363,1 cc. So R.Q. in 3^b = 0,87.

Curve 3^c fall in 10 min. = 0 mm., so CO₂-output = O₂ consumption. So R.Q. in 3^c = 1.

Curve 3^d Rise in 10 min. = 1,7 mm.; consequently CO₂-output = 379,9 cc.

Curve 4. Fall in 10 min. 28,2 mm. consequently O₂-consumption = 358,0 cc.

When assuming that during curve 3^d, traced about 10 minutes earlier, there was an O₂-consumption equal to the one during curve 4, then the R.Q. during the period of curve 3^d has been $\frac{379,9}{358,0} = 1,06$.

In this experiment we observe, therefore, that after the introduction of insulin a progressively increasing rise of the R. Q. reveals itself from 0.75 upwards to 0.87, and from 1 to 1.06. This increase

of the R. Q. appears distinctly from the curves of fig. 2, while we can also infer from them that hyperventilation is not answerable for this increase, neither in this experiment, nor, indeed, in any of the others. This will be readily understood, if we consider that from curve 3^a, just before the insulin-injection, we register a ventilation of 1095 cc. per minute and from 3^d, after insulin, a ventilation of 1140 cc. per minute, that is an increase of only 4 %. That this inappreciable hyperventilation should be answerable for the rather prolonged rise of the R. Q. is in itself not admissible; moreover this view is disproved directly by the fact illustrated by curve 3^a, that the much more pronounced increase of ventilation, occurring immediately after the insulin-injection, does not involve the least change in the course of CO₂-curve.

Besides, in this stage of the experiment there was also a decrease in the bloodsugar-content from 4 ‰ (hyperglycemia after decerebration) down to 1.3 ‰¹⁾. The O₂-consumption displayed a fall from 415.9 cc. per kg.-hour to 358.0 cc. per kg.-hour.

That the rise of the R. Q. under the influence of insulin cannot possibly be attributed to hyperventilation is proved additionally by our experiments, which yielded curves that showed an increase of the R. Q. even with smaller ventilation than existed before the insulin-injection.

The rise of the R. Q. generally begins 20 min. after the injection of the insulin, and is most often gradual, as in the experiment recorded here, sometimes, however, it is rather abrupt.

Of course, the principal interest centres in the question how is the increase of the R. Q. to be explained? So long as the R. Q. does not rise above 1, this increase may be assumed to have evolved from the increased combustion of carbohydrates under the influence of insulin. This view has been adopted by BANTING, BEST, COLLIP, HEPBURN and MACLEOD, when they found that insulin raises the R. Q. in the pancreas-diabetic dog and in the normal dog. Other researchers, e.g. KROGH and DALE have also accepted this view. However, this is presumably not the only factor in this process, as is borne out by those cases in which the R. Q. rises above 1 under the influence of insulin. For the present these cases are perhaps best interpreted when assuming, in accordance with MACLEOD's and GEELMUYDEN's opinion, that under the influence of the pancreas hormone also glycose is converted into substances that contain less of O₂ in their

¹⁾ In a later phase of the experiment the well-known hypoglycemic phenomena revealed themselves, from which we conclude that, in the stage of the experiment recorded here, the insulin-action had not yet culminated.

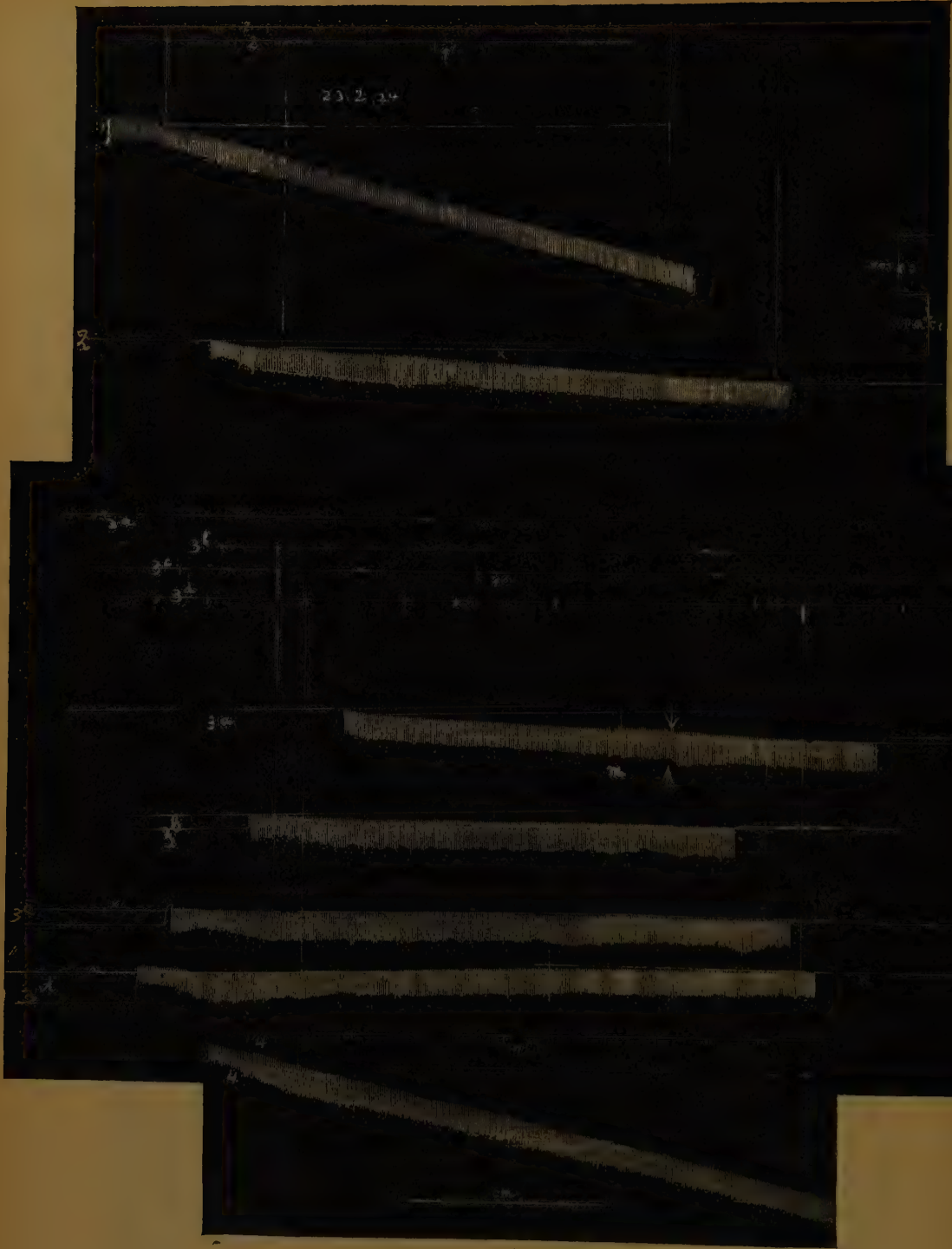


Fig. 2.

Curve 1 and 4: 2O_2 -curves, 1 *before*, curve 4 *after* insulin.

Curve 2: CO_2 -curve *before*, curves 3^b , 3^c and 3^d *after* insulin.

In curve 3^a at $\downarrow$ intravenous injection of 4 units Iletin (U_{20}); at m beginning of the manipulations for the intravenous injection; the lines x in the curves 2, 3^a , 3^b etc. parallel to the control-lines of the "trap"-system, are used to measure in the middle of the curves the extent of the fall, resp. the rise.

Under curve 4 5 cm. illustrate the magnitude of the diminution. The time-lines, given for each of the curves separately, because the kymograph was not absolutely regular, indicate minute-intervals. By the side of curves 1 and 2 calibration of the spirometer (200 cc. = 29 mm.).

molecule, e.g. into fat. This assumption tallies with the long known fact that animals, fattened on a carbohydrate-containing diet, have a R.Q. higher than 1. Of course it is highly presumable that this factor exists also in those cases, in which the R.Q. does not rise above 1, but that it is superseded there by other factors, or performs an inferior function. Doubtless, the action of insulin is a very complicated problem, and still far from being set at rest. However, these experiments have shown:

1°. *that under the influence of insulin (Iletin Lilly and Insulinum Neerlandicum) in the decerebrate fasting cat mostly (in 5 out of 6 cases) a distinct rise of the R.Q. appears, which cannot be ascribed to hyperventilation.*

2°. *that in some cases (2 out of 5) the R.Q. rises even above 1.*

This method of graphical registration of the CO_2 -output with the aid of our "CO₂-trap" is peculiarly adapted for similar researches.

March, 1924. *From the Physiological Laboratory
of the University of Utrecht.*

Physics. — "*The Crystal Structure of Mercuric Sulphide. I.*"

By N. H. KOLKMEIJER, J. M. BIJVOET and A. KARSEN. (Communication Nr. 14 from the Laboratory of Physics and Physical Chemistry of the Veterinary College). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of May 3, 1924).

§ 1. *Introduction.* Some modifications are known of HgS ; so far as we know only one of these — the hexagonal cinnabar¹⁾ — has been investigated with X rays. We intend to examine the different black and red modifications mentioned in the literature²⁾. Here we give the results regarding the so called amorphous black HgS ; in this form mercury is precipitated out of solution by a sulphide.

§ 2. *The apparatus.* The apparatus was of the ordinary type used in the powder method³⁾. A tin diaphragm, length 60 mm. diameter 2 mm., reduced the black central spot. Besides, as in the experiments of DE SMEDT and KEESOM⁴⁾, a little tube of calculated dimensions formed the lengthening piece of the diaphragm in the camera in order to protect the film from the secondary rays excited at the diaphragm. The radius of the camera was 27,2 mm. The sample was in a thinwalled glasstube of 1 mm. diameter. By a Ni -filter all but the $CuK\alpha$ -rays were for the greater part absorbed.

§ 3. *Röntgenograms and Calculation.* In columns 1 and 2 of the table are recorded the distances on the film from the middle of the image to the interference lines and the estimated intensities; in column 3 these values are diminished by $\frac{r}{2}(1 + \cos \vartheta)$, the approximate correction for the thickness of the rod according to GERLACH and PAULI⁵⁾. Column 4 contains the corresponding values of $10^3 \cdot \sin^2 \frac{\vartheta}{2}$

¹⁾ CH. MAUGUIN, C.R. 176, 1483, 1923.

²⁾ I.a. E. T. ALLEN and J. L. CRENSHAW, Zs. anorg. Chem. 79, 155, 1913.

³⁾ Cf. A. J. BIJL and N. H. KOLKMEIJER, these Communications Nr. 1, these Proceedings 21, 405, 1918.

⁴⁾ J. DE SMEDT and W. H. KEESOM, these Communications Nr. 10, these Proceedings 25, 118, 1922.

⁵⁾ W. GERLACH and O. PAULI, Zs. f. Physik 7, 116, 1921.

showing the common factor 17.35. Column 5 gives the values of Σh^2 following from this factor and column 6 the calculated values of $10^3 \cdot \sin^2 \frac{\theta}{2}$. The agreement with the observed values of column 4

is very good. Column 7 gives the indices of the reflecting planes.

Therefore the "amorphous" HgS crystallizes in the cubic system.

From the mentioned value of the common factor the lattice parameter is found to be

$$a = 5.85 \cdot 10^{-8} \text{ cm.}$$

In combination with the value of the density this gives about 4 for the number n of HgS-groups per unit-cell. Starting from $n = 4$ we find for the density

$$d = 7.69$$

in good agreement with the highest of the values given in LANDOLT-BÖRNSTEIN-ROTH 1912 (7.55—7.70).

"AMORPHOUS" HgS.							
Distances in mm.	Intensity estim	Distanc. corr.	$10^3 \cdot \sin^2 \frac{\theta}{2}$ observed	Σh^2	$10^3 \cdot \sin^2 \frac{\theta}{2}$ calculat.	$h_1 h_2 h_3$	Intensities calculated
1	2	3	4	5	6	7	8
13,3	vs	12,6	53	3	52	111	6
15,0	w	14,5	67	4	69	200	2
21,3	vs —	20,9	141	8	139	220	12
24,9	vs	24,5	190	11	191	311	17
26,2	vw	25,8	209	12	208	222	3
30,5	vw	30,1	277	16	278	400	6
33,6	m	33,3	330	19	330	331	17
34,6	wm	34,3	348	20	347	420	10
38,5	m	38,2	418	24	416	422	24
41,4	m	41,1	470	27	468	$\left. \begin{matrix} 333 \\ 511 \end{matrix} \right\}$	23
45,9	w —	45,7	554	32	555	440	12
48,6	s	48,4	603	35	607	531	34
49,9	w	49,7	626	36	624	$\left. \begin{matrix} 442 \\ 600 \end{matrix} \right\}$	12

The positions of the particles in the cell are yet to be determined by the consideration of the intensities of diffraction. Column 7 shows that the mixed indices planes are absent, from which it is obvious that the *Hg*- and the *S*-particles are arranged in face-centered cubes. Now two relative positions of these lattices are possible (NaCl- and ZnS-structure resp.); of these only the latter (space group T_d^2 ; relative displacement along the diagonal over a quarter of its length) is adequate. In calculating the intensities all the continuous factors are suppressed, the intensity I being taken proportional to $\nu |S|^2$; here ν is the number of planes-factor and

$$S = (1 + e^{\pi i(h_2+h_3)} + e^{\pi i(h_3+h_1)} + e^{\pi i(h_1+h_2)}) (Hg + Se^{\pi i(h_1+h_2+h_3)})$$

where $Hg = 78$ (number of electrons in the *Hg*-ion) and $S = 18$ (idem in the *S*-ion). So only the observed and calculated intensity ratios of neighbouring lines are to be compared; columns 2 and 8 show very good agreement.

From the given structure the sum of the radii of the atomic domains of *Hg* and *S* follows to be 2,53 Å; according to BRAGG the radius of the atomic domain of *S* amounts to 1,02 Å; the radius of *Hg*, which, so far as we know, was not till yet determined, then appears to be 1,51 Å.

§ 4. Summary.

Black precipitated *HgS* is of *ZnS* structure. Lattice parameter $5.85 \cdot 10^{-8}$ cm. Density 7.69, radius of the atomic domain of *Hg* $1.51 \cdot 10^{-8}$ cm.

Utrecht
Amsterdam } , April 4, 1924.

Chemistry. — "*The Mass-Ratio of Isotopes in Chemical Elements.*"

By Prof. F. M. JAEGER and Dr. D. W. DIJKSTRA.

(Communicated at the meeting of May 3, 1924).

Since ASTON, by an ingenious modification of J. J. THOMSON'S method of positive rays, was able to prove the complex nature also of a number of non-radioactive elements, — the question has obtained renewed interest for the chemist, as to what significance should really be attributed to the ordinary atomic weights of the elements, as finally determined after the accurate investigations during more than a century. More particularly there is reason to ask, whether the atomic weights of the common, non-radioactive elements can any longer be considered as real constants or not? For in the case of some elements, isotopes of certain products of radioactive desintegration, it has been proved on the one side, that their apparent atomic weights were different, according to the special nature of the radioactive minerals from which they were separated, just because of the unequal ratio in which in each case the isotopous desintegrations-products, no longer separable by chemical means, were present in their mixture. On the other hand it followed from ASTON'S investigations, that the greater number of the ordinary elements studied by him, consists of isotopous components, the atomic weights of these being represented by whole numbers, if calculated with respect to that of oxygenium = 16. The values of the observed atomic weights of the chemical elements, — which in general deviate from whole numbers, — must, therefore, depend on the special ratio in which the isotopous components are present in the mixture, formerly considered to be homogeneous in nature. If under different circumstances this mixing ratio should also appear to be variable, then the atomic weights of the common elements, as determined in the usual way, could no longer be considered as being true constants of nature.

However, experiments up till now seem to indicate, that no such variability of the ordinary atomic weights can be observed. At this moment a great number of values of atomic weights are accurately known as a result of the very exact atomic weight-determinations of the last thirty years. Within the limits of the inevitable experimental

errors, those values appeared always to be identical. From this fact it seems safe to draw the conclusion, that the mixing ratios of the isotopous components with their whole-numbered atomic weights, are in ordinary elements under all circumstances the same and invariable. It is, however, not so easy to answer the question *why* this is the case. For it appears that it is possible to consider this question from more than a single point of view.

It may be supposed, — and this is the oldest point of view, accepted originally by physicists, — that the atomic weights appear therefore always to have the same values, because the materials studied in reality represent essentially the same objects. For in the course of the enormous geological ages, measuring hundreds of millions of years, that have elapsed since the genetic process of the elements, — all matter in the solar system has been so thoroughly mixed up, that e.g. all local differences which eventually might have once existed on the earth, have long since become effaced. Each investigated element, therefore, independent of its locality on earth, may thus be considered as being in all cases taken from the same and completely homogeneous main store.

On the contrary, also another point of view may be adopted: it may be supposed that a more intrinsic cause, connected with the nature of the atoms themselves, is responsible for the apparently constant ratio of the isotopous kinds of atoms in the so-called chemical elements, instead of being that cause attributed solely to the homogenizing influence of external circumstances on matter. The case might be, that the said constant ratio might depend on the degree of relative mechanical or electromagnetic stability of the very complex moving systems we call atoms, it being in a certain way the expression of the equilibrium-ratio determined by the relative degree of probability of the different atomic systems, as it existed at the moment when those systems took their origin simultaneously from the chaotic mass of irregularly moving protons and electrons by the primordial process of differentiation. If — let us say — *two* kinds of atomic systems with equal nuclear charges, but different nuclear masses, originated in that process, the number of atoms of each kind then produced would evidently depend on the relative probability or stability of each of them under those circumstances. The fact that in our common elements the mass-ratio of two isotopous components is often strongly shifted towards one of them, so that the atoms of that kind are present in the mixture in a preponderant quantity, — may be considered as an indication, that really their probability during the genetic process was not altogether the same.

Moreover, it is probable that external circumstances only played a secondary rôle during that process: at least experience teaches us, that also in other respects the variation of external parameters does not influence appreciably the special rate or character of such intratomic phenomena.

Finally a third supposition has been made by some authors, namely, that the mixtures of isotopes we call „elements”, are residues or products of former radioactive desintegrations, just as this is the case for the “lead”-species of variable atomic weights met with in radioactive minerals; be it that these mixtures are really inactive final products of such desintegration-processes, which are very slowly transformed with accompanying radiation, or without such radiation; or that they are mixtures whose radioactivity cannot be demonstrated by our actual means. In the last case it might be possible that, as in the case of the “lead”-species mentioned, a difference in atomic weight could be found, dependent on the former history of these products. We could e.g. imagine, that differences in atomic weight were demonstrable by sufficiently exact methods, if the elements considered originated from different geological formations or from different localities on earth.

In later years the opinion with respect to the problem under consideration has gradually gone in the direction of the second supposition mentioned above, i.e. of that, according to which an intrinsic, statistical cause should exist which determined the special ratio of the isotopes, as found in the common elements; this cause being then related to certain factors of stability of the different atomic systems. This view would be strongly sustained, if it were possible to prove the identity of the ratio of the isotopous components also for an element of cosmic origin, whose generative circumstances very probably have been quite different than for the same element on earth.

The only way for such an investigation is afforded by the study of that cosmic material which reaches the earth as *meteorites*. Leaving for the moment aside the question, whether these meteorites always formed a part of the solar system or if they came from the outside of it, — it must, however, appear certain, that the generative circumstances of those cosmic masses were completely different from those of terrestrial aggregates of analogous chemical constitution. More especially is this the case with that class of meteorites, which ordinarily is discerned as that of the *aerolites* or *chondrites*. For the greater part they consist of compounds of *siliciumdioxide* with numerous bases, as e.g. those of *calcium*, *magnesium*, *iron*, *alu-*

minium, etc. They occur in aggregates which, as follows from their petrographic character, originated under circumstances totally different from such as were present in the formation of terrestrial rocks of analogous chemical constitution. From ASTON's work it follows, that *silicium* consists of *two* isotopes, whose atomic weights are **28** and **29**; the atomic weight of ordinary *silicium* being higher than **28** and lower than **29**, indicates a mixture of definite ratio of both components, which appears as a chemically homogeneous substance.

It has been our purpose to determine with the utmost accuracy the ratio of both isotopes in terrestrial *silicium* of different origin and to compare it with the same ratio in *silicium* of cosmic origin. Perhaps it would be possible in this way to state a difference in the ratio of both isotopes in these different kinds of *siliciumdioxide*. In accordance with this we investigated in the way described further on the *silicium* from a number of terrestrial minerals of very different local and geological origin and afterwards from a number of chondrites of different petrographical character and of widely deviating times and localities of fall, so that the probability was increased as much as possible to find amongst those objects material of different cosmic origin. The results of these measurements, executed with the highest accuracy momentarily attainable were all obtained in this laboratory and will be published within a short time in all details in the dissertation of Mr. DIJKSTRA. Therefore a concise review of the method applied and the chief results obtained must suffice here.

The method which in the first place presents itself here would be a direct atomic weight-determination of the *silicium* separated from all these silicates. However, this method could not be followed here, because the atomic weight of the element *silicium* is one of the least accurately determined values of this kind. The data available in literature oscillate between 28,06 and 28,39, the cause of which are the extraordinary difficulties met with in the exact atomic weight-determination as a consequence of the particular properties of the *silicium*-compounds suited for this purpose. But even the best and most accurate atomic weight-determinations would not afford a degree of accuracy great enough for the purpose desired in this case, so that even then it would not be allowable to deduce with any certainty the very small deviations of the ratio of the isotopes in mixtures of them, where the atomic weights of both components do not differ more than a single unit. Disregarding here the possibility of the existence of a third *silicium*-isotope (with a mass = 30), the measured atomic weight would correspond with a mixture con-

taining 6% *Si* of the heavier and 94% of the other kind, while the atomic weight-determination of 28,39 would on the other hand correspond with a mixture of 39% *Si* of the mass 29 and 61% *Si* of the mass 28. From this it becomes evident that an error in the atomic weight-determination of only 0,1% would cause an apparent shift of the ratio considered of no less than 3%.

Therefore we have used a fact which enables us to reach a much greater accuracy. The chemical properties of isotopous elements, only depending on the outer electrons, are rigorously identical. If a certain ratio of them is, therefore, present in the mixture we call the element *silicium*, it will be clear that, independent of what chemical reactions are executed with it, that ratio will remain unchanged in all derivatives prepared from it. Moreover, the specific weight of isotopous elements is directly proportional to their atomic weight. The fact is most probable, because they occupy the same place in the periodical system of elements, having, therefore, identical atomic volumes, and from this the proportionality between atomic and specific weights follows immediately. That this view is a sound one was proved by experiment by the accurate determination of SODDY and HYMAN¹⁾ with *thorite*-"lead" and by those of RICHARDS²⁾ with isotopous mixtures of "lead" from different radioactive minerals: both atomic and specific weights of these mixtures were determined and a rigorous proportionality between both constants was found in all cases. The same is true also for the *derivatives* of such elements, as was demonstrated for the *nitrates* of these "lead"-species by FAJANS and LEMBERT³⁾.

Every real deviation of the specific weight of isotopes or of that of the compounds prepared from it, must thus necessarily correspond with a real shift in the ratio of the isotopous components in the mixture. Much more exactly than by a direct atomic weight-determination, the ratio considered can be determined by comparison of the specific weights of the same derivative of *silicium*, if measured with the highest attainable accuracy.

To fix the choice of a compound suited for this purpose, the question arose to find a perfectly stable liquid substance, which

¹⁾ F. SODDY and H. HYMAN, *Trans. Chem. Soc.* **105**, 1402, (1914). The specific weight of *thorite*-"lead" was 0,26% higher than that of ordinary lead; but at the same time the atomic weight as determined by SODDY (207,69) and by HÖNIGSCHMID (207,77) was in agreement with the mentioned supposition, which involves in this case a value of 207,74.

²⁾ TH. W. RICHARDS, *Pres. Address Amer. Assoc. Adv. of Science*.

³⁾ K. FAJANS and M. LEMBERT, *Zeits. f. anorg. Chem.*, **95**, 297, (1916).

could be obtained in a completely pure state and which possesses such properties, e. g. of unattackability, that it might be used for these measurements without objections. Finally we chose *tetra-ethyl-silicane*: $\text{Si}(\text{C}_2\text{H}_5)_4$, a colourless liquid with a boilingpoint of 154°C ., which in many respects shows a character like that of the *paraffines*. It is highly stable and may e. g. be shaken during many days with concentrated sulphuric acid, without showing any trace of decomposition. Moreover, it follows from ASTON's experiments, that *carbon* and *hydrogen* are both "full-bred" elements, without isotopous components, so that the whole effect eventually to be found must be wholly ascribed to a change in the ratio of the *silicium*-components alone. We preferred this compound to the corresponding *tetramethyl-silicane*, because this last has a boilingpoint some 120° lower than the first, so that it is much more volatile and might, therefore, more easily cause errors in the pycnometric determinations, while on the other hand a fractionating within a few hundreds of a degree may be executed with less difficulty in the case of the higher boiling liquid.

The preparation of the *tetra-ethyl-silicane* from the different kinds of *siliciumdioxide* was made as follows. The *siliciumdioxide* required was separated from the minerals and chondrites by melting the finely powdered silicates in platinum crucibles, carefully and in small quantities to avoid explosions in the case of the chondrites, with an excess of pure *sodiumcarbonate* and *sodiumnitrate*, heating the product with *hydrochloric acid* and repeating this treatment several times with the filtered, washed and dried *siliciumdioxide*. Particularly in the case of the chondrites, this operation must be executed with great care, because they contain *schreibersite* (*ironphosphide*), *pyrrhotine*, *troilite* and other *sulphides*; the platinum-crucibles are strongly attached therefore in these cases. Finally the *siliciumdioxide* was obtained in this way in the form of a pure white, voluminous powder, which was converted into the desired compound in the following way.

The *dioxide* was transformed in a silver dish by means of a solution of *ammonium-fluoride* carefully freed from all *siliciumdioxide*, into *ammoniumhexafluosilicate*: $(\text{NH}_4)_2\text{SiF}_6$ by addition of strong *hydrochloric acid*. This salt was converted into the corresponding *barium-salt*, which on heating is decomposed into *barium-fluoride* and *siliciumtetrafluoride*: SiF_4 . This gas was led in a perfectly pure and dry state into a perfectly dry ethereal solution of a great excess of GRIGNARD's reagents, in which *magnesium-ethyl-bromide* was the active substance. All possible care was taken to prevent any contamination of the product; finally the substance was

obtained in a perfectly pure state as a colourless liquid, boiling at a pressure of 760 mM. at $154^{\circ},95$ C. Analysis gave: 66,48 % C and 14,54 % H; calculated: 66,49 % C and 13,97 % H. A molecular weight-determination showed, that the molecular weight really corresponded with that of $\text{Si}(\text{C}_2\text{H}_5)_4$.

In this way *twelve* different preparations were obtained: *six* from *terrestrial* silicates of strongly deviating origin, namely from *quartz-sand* from *East Groningen*, from a *German* and a *North American* locality, further from *analcite* (*Seisser Alp*), from *leucite* (*Tavolato near Frascati*) and from *chlorite* (*Pfitschtal*). *Analcite* is a zeolithic silicate, *leucite* a volcanic product, *chlorite* a silicate of metamorphic origin.

Six other preparations were obtained from *siliciundioxyde* of *cosmic* origin: from an *aërolite* of *Alfianello* in *Italy* (16th february 1883), of *Soko Banya* in *Servia* (13th october 1872), of *Holbrook* in *Arizona* (19th july 1912), of *Ochansk* in *Russia* (*Perm*) (30th august 1887), of *Bjurböle* in *Finnland* (12th march 1899) and of *Kesen* in *Japan* (13th june 1850). Of these chondrites that from *Soko Banya* is a *peridotite-enstatite-chondrite* with fully 40 % SiO_2 ; that from *Ochansk* a *hypersthene-olivine-chondrite* with 37 % SiO_2 ; that from *Holbrook* a typical *enstatite-chondrite* with 30,8 % SiO_2 ; the chondrite of *Bjurböle* had 41 % SiO_2 . Besides *pyrrhotine*, *troilite* and *schreibersite*, small granules of *iron-nickel* and *chromite* were found; some chondrites contained also traces of *copper*, probably in the form of sulphur-compounds. In Fig. 1—3 some photographs are reproduced of microscopical sections in ordinary and polarized light; the black inclusions consist of *troilite*, *chromite* and *iron-nickel*. The powder of these chondrites is pale ash-grey to greyish-brown; they possess a trass-like, finely granular structure.

The pycnometer used had a volume of about 3 cm³. and was of the wellknown OSTWALD-SPRENGEL-type, slightly modified and made of a special kind of hard glass for the purpose of preventing a change of the glass-surface by the water of the thermostate. The temperature of the thermostate could be kept constant within $0^{\circ}.01$ C. and was perfectly constant (within $0^{\circ}.001$ C.) during the short time, that the pycnometer was immersed. To prevent as much as possible the correction for the mercury-column above the water-surface, the thermometer was circumflowed totally by the water of the thermostate and so read, that no incertitude by parallaxis was introduced. All weighings were executed by means of a micro-balance, which was enclosed in a box of red copper to protect it against temperature-radiations. Instead of by the ordinary mirror-readings, the

oscillations of its beam were measured at a considerable distance by means of a telescope and a movable lighting spot, obtained by means of a reflected light-beam in a small mirror fixed in the middle of the beam of the instrument and a prism-arrangement. All weights were calibrated, as in atomic weight-determinations and reduced on the vacuum; the same was the case with all weighings, while the corrections for the upward pressure were made as small as possible by applying a counter-weight of about the same volume as the pycnometer had. No measurements were made before the pycnometer had hung in the balance-box for three quarters of an hour, when it completely had accepted the temperature of its environment. All radiations from the human body were prevented in this way, because all readings were made at a considerable distance from the instrument. Arrangements were made to prevent the occurrence of electrical charges at the agate supports and at the walls of the balance which might cause a shift of the zero-point, by placing some radioactive *thorium-carbonate* in the box of the instrument; by preliminary experiments all corrections were moreover exactly determined with respect to the inequality of the balance-beams, the changes of atmospheric pressure, of temperature, etc. To give an impression of the high degree of reproducibility of these measurements acquired, it may be mentioned, that in subsequent weighings of a mass of about totally 10,5 grammes, identical results were obtained within a few unities of the *sixth* decimals, after all necessary corrections had been made: hundredth parts of milligrammes can thus be determined with perfect certainty and only the thousands parts possess less significance. Originally some deviations occurred by the filling of the pycnometer as a consequence of capillar and other influences; after a modified construction of the apparatus and another kind of glass being chosen, this difficulty also was finally overcome.

By far the greatest difficulties were met with during this investigation in the purification of the different preparations. In the GRIGNARD-reaction a number of by-products take their origin, chiefly some *ethyl*-derivatives and among them *tri-ethylsilicol* (C_3H_7), $SiOH$, the boilingpoint of which differs only slightly from that of *tetra-ethyl-silicane*, being therefore only with great difficulty separable from it by fractional distillation. Complete purification appeared only possible by repeating the GRIGNARD-reaction several times, by reduction of still present *silicol* and decomposition of the also formed *triethylsilicol-ethylether* by heating with *hydrojodic acid* during several days, shaking for a long time with concentrated *sulphuric acid* and

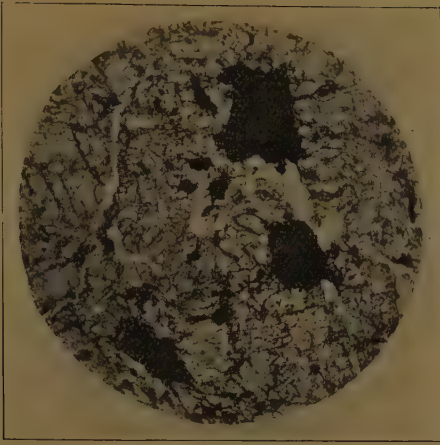


Fig. 1a. Chondrite of *Alfianello*.
in ordinary light.

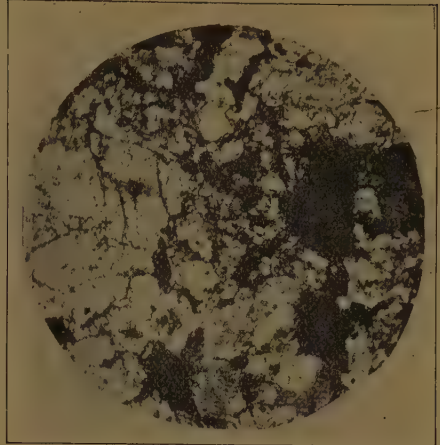


Fig. 1b. Chondrite of *Alfianello*.
between crossed nicols.

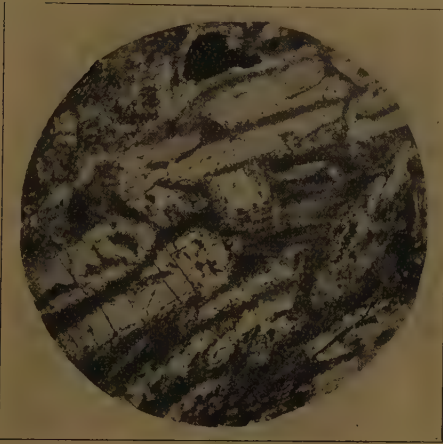


Fig. 2a. Chondrite of *Soko Banya*.
in ordinary light.

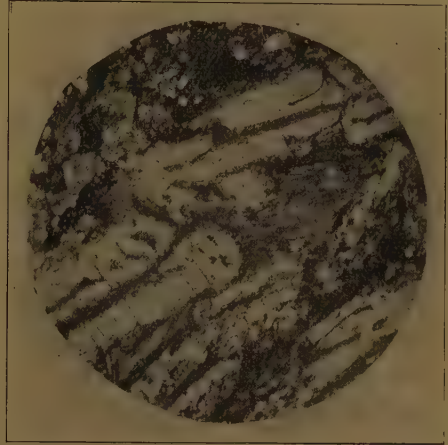


Fig. 2b. Chondrite of *Soko Banya*.
between crossed nicols.

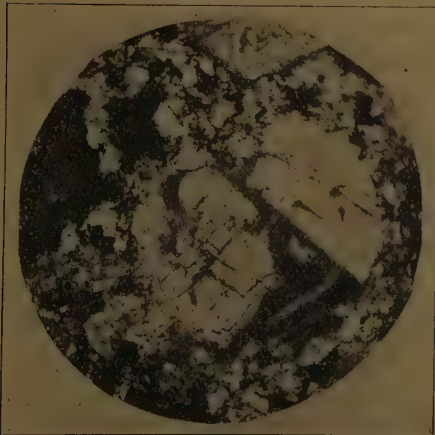


Fig. 3a. Chondrite of *Ochansk*.
in ordinary light.

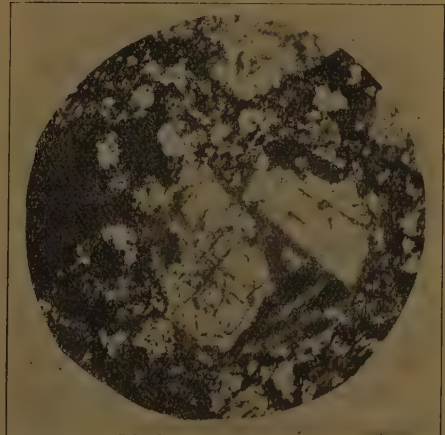


Fig. 3b. Chondrite of *Ochansk*.
between crossed nicols.

repeated distillation over *phosphorus-pentoxide* and fresh metallic *sodium*. Finally, the liquid obtained was subjected to fractional distillation under constant pressure and, after calculation of the true boiling-point from the vapour-tension-curve, only that fraction was separated, which under the given pressure boiled within a few hundredths of a degree centigrade. The liquid was not considered to be sufficiently pure, before no deviations of the specific weight in subsequent measurements could be stated any longer. Then the final determinations were made with this liquid. The preparations thus obtained might be considered as "chemically pure" in the common sense of the word, so that e.g. no differences in refractive index or in chemical behaviour could be stated any more. The numerous details of the technical side of these experiments may be seen from the larger publication mentioned above.

In the table on page 403 the results of these measurements are reviewed.

From this it is evident that the specific weights at 20° C. of the *tetra-ethyl-silicane* of *terrestrial* origin, independent of the fact whether the *siliciumdioxide* used came from *quartz-sand* of different localities or from silicates of zeolithic, vulcanic or metamorphic character, do not differ from each other more than 0,006 % of the main value: $0,76734^s \pm 0,00005$; the deviations are irregular to both sides. For the *cosmic* preparations the main value is: $0,76735 \pm 0,00003$; the deviations are not greater than $\pm 0,005$ %. The differences between the determined values with terrestrial and cosmic material are not greater than about 0,002 to 0,003 %.

The values indicated with Δ are the variations in main atomic weight, as they may be calculated from the observed specific weights, supposing that their differences were really caused by a different ratio of the isotopous components of *silicium*. It appears that the main atomic weights, even then, could deviate no more than a few unities of the *third* decimal; these deviations would, therefore, be far beyond the attainable accuracy of the best method of atomic weight-determination, as was already emphasised in the above.

However, it must now again be asked, what is the significance of these deviations found in the different preparations? For, although only very small, they are real, because they are much greater than the experimental error of the method followed. The method allows an accuracy of measurement of about 0,0004 %, so that the deviations mentioned are in every case ten times as great.

Experience gained during this work compels us to conclude, that these differences are connected with the presence of extremely

minute traces of by-products, admixed with the preparations of *tetra-ethyl-silicane*: traces so small, that they cannot be indicated any longer by any reagent, and that they cannot influence in any way the chemical character of the different substances.

To get more information about this and greater certainty, we

T A B L E.

A. Terrestrial material:	Boilingpoints of fraction:	Theoret. boiling-point:	Weight of Pycnometer:	Specific ¹⁾ weight at 20° C.	($n-n'$):	Δ
Netherlands (Groningen)	$p=748,8; 154^{\circ},10-154^{\circ},15$	$154^{\circ},1$	2,27760	0,76730	+0,000026	-0,0095
Germany	$p=756,6; 154^{\circ},50-154^{\circ},54$	$154^{\circ},5$	2,27766	0,76732	+0,000024	-0,0057
N. America (Penns.)	$p=765,8; 154^{\circ},84-155^{\circ},0$	$155^{\circ},0$	2,27761	0,76730	+0,000082	-0,0089
Analcite (Seisser Alp)	$p=749,0; 154^{\circ},16-154^{\circ},20$	$154^{\circ},1$	2,27790	0,76740	-0,000005	+0,0095
Leucite (Tavolato, Frascati)	$p=765,5; 154^{\circ},98-155^{\circ},06$	$155^{\circ},0$	2,27784	0,76738	-0,000002	+0,0057
Chlorite (Pfätschtal)	$p=756,2; 154^{\circ},46-154^{\circ},60$	$154^{\circ},5$	2,27778	0,76736	+0,000005	+0,0019
B. Cosmic material:						
Chondrite of Alfianello (Italia) 16. 10. 1883.	$p=760,8; 154^{\circ},62-154^{\circ},70$	$154^{\circ},7$	2,27774	0,76735	+0,000018	-0,0006
Chondrite of Soko Banya (Servia) 13. 10. 1872.	$p=763,4; 154^{\circ},84-154^{\circ},90$	$154^{\circ},9$	2,27776	0,76735	+0,000012	+0,0006
Chondrite of Holbrook (Arizona) 19. 7. 1912.	$p=759,8; 154^{\circ},66-154^{\circ},70$	$154^{\circ},7$	2,27776	0,76735	+0,000013	+0,0006
Chondrite of Ochansk (Perm.) 30. 8. 1887.	$p=760,6; 154^{\circ},65-154^{\circ},68$	$154^{\circ},7$	2,27784	0,76738	—	+0,0057
Chondrite of Björbole (Finland) 12. 3. 1899.	$p=765,2; 154^{\circ},92-154^{\circ},96$	$154^{\circ},9$	2,27775	0,76735	+0,000002	0
Chondrite of Kesen (Japan)	$p=753,9; 154^{\circ},35-154^{\circ},50$	$154^{\circ},4$	2,27776	0,76735	+0,000001	+0,0006

¹⁾ The real temperature was $19^{\circ},93^{\circ}$ C.; in abbreviation 20° C. is written.

have determined the differences of the refractive indices for sodium-light between the different preparations by means of the interferometer of LOEWE, as constructed by ZEISS. If the deviations found were really caused by a variation of the mixing ratio of the *silicium*-isotopes, there could not be any difference between the refractive indices, because the optical properties of isotopous elements for wavelengths within the limits of the visible spectrum are identical, as being again only a function of the outer electrons of the atoms. But if these deviations were connected with the presence of minimal traces of impurities, there would be a good chance, that the interferometric method would really reveal such differences in the refractive indices. In that case, however, a rigorous proportionality between the deviations of the specific weights and those of the refractive indices could, of course, not be expected: there are, as we have seen, several kinds of impurities possible here, and they will be simultaneously present in variable and unknown quantities in each of the preparations. At the best there could be expected a kind of rough anti-parallelism¹⁾ between both series of numbers.

The measurements were executed in such a way, that as the liquid of comparison in all cases, the *tetra-ethyl-silicane* from the meteorite of *Ochansk* was always taken in one of the two vessels of the chamber of the instrument. The deviations of the refractive indices of the other preparations with respect to the liquid mentioned, are given in the Table under $(n-n')$; the division of the drum of the micrometer is such, that on using this special chamber, each division corresponds to two units of the *sixth* decimal of the refractive index.

If now the *tetra-ethyl-silicane*-preparations were arranged in the same way as in the Table and the values of the specific weights or of the numbers $(n-n')$ were taken as ordinates in a graphic figure (Fig. 4), two broken lines were obtained of the form represented therein; the numbers $(n-n')$ are taken with opposite algebraic signs. Although evidently a strong proportionality is not present, nobody can doubt any longer the striking parallelism between both series of values. From this it must be considered as highly probable, that the deviations of the specific weights of the *tetra-ethyl-silicane*-preparations have no real significance, as far as regards the eventual difference in ratio of the *silicium*-isotopes in them: for in that case the values $(n-n')$ must necessarily be all equal to zero. Within the limits of the

¹⁾ *Anti-parallelism*, because the specific weight of the preparations is lower, as they are purer. Therefore the algebraic signs of $(n-n')$ are inversed in Fig. 4.

highest accuracy at this moment attainable, the specific weights, and, therefore, the said ratios of the isotopes, must consequently be

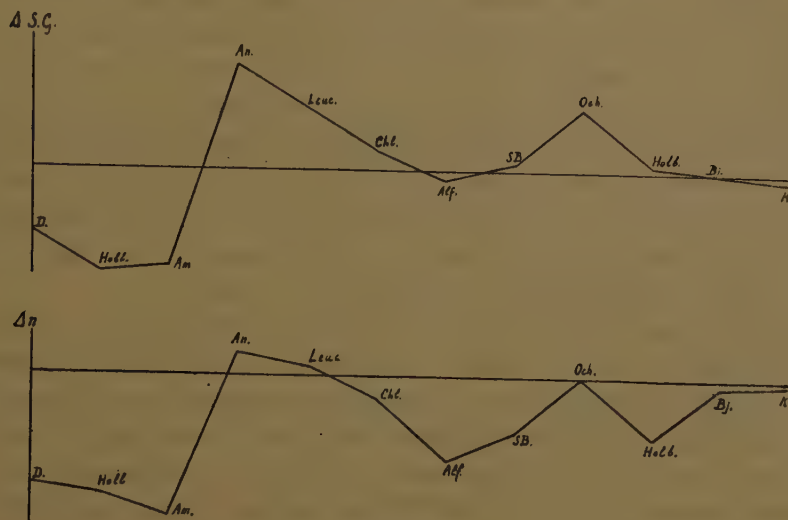


Fig. 4.

identical as well in the case of the terrestrial as of the cosmic silicon-species.

In connection with these results attention may finally be drawn to the analogous conclusions, to which other investigators before us have come in this respect, — even although to these results, with a single exception perhaps, no such high degree of accuracy can be attributed as to ours, as a consequence of the methods applied.

BAXTER and THORVALDSEN, HOOVER, PARSONS and HILTON¹⁾ found the atomic weights of *iron* and of *nickel* from the *Cumpas*-meteorite identical with the atomic weights of the terrestrial elements within the limits of experimental errors. I. CURIE²⁾ compared the corresponding value for *chlorine* from seawater, from a Canadian *sodalite* and from *apatite* by weighing of *silverchloride*; GLEDITSCH, SAMDAHL³⁾ and DORENFELDT⁴⁾ proved the same identity for *chlorine* from a Norwegian and another specimen of *apatite*; BRÖNSTED and VON

¹⁾ G. P. BAXTER and T. THORVALDSEN, Journ. Am. Chem. Soc., **33**, 337, (1911).

G. P. BAXTER and C. R. HOOVER, Journ. Amer. Chem. Soc., **34**, 1657, (1912).

G. P. BAXTER and L. W. PARSONS, Journ. Amer. Chem. Soc., **43**, 507, (1921).

G. P. BAXTER and F. A. HILTON, Journ. Amer. Chem. Soc., **45**, 694, (1923).

²⁾ I. CURIE, Compt. rend. Paris, **172**, 1025, (1921).

³⁾ E. GLEDITSCH et M. B. SAMDAHL, Compt. rend. Paris, **174**, 746, (1922).

⁴⁾ M. DORENFELDT, Journ. Amer. Chem. Soc., **45**, 1577, (1923).

HEVESY ¹⁾ found by accurate measurement of the specific weight of *mercury* of nine different localities values, which did not differ 0,1 %; RICHARDS ²⁾ did neither succeed in finding a difference in the atomic weights of *copper*, and of the *calcium* of *calciumcarbonate* and the *chlorine* of *sodiumchloride* from different localities; MONRO ³⁾ also came to the same conclusion in his measurements of *borium* (titration-method). On the contrary, MUZAFFAR ⁴⁾ mentioned differences of 1 to 1,5 % in the atomic weight of *antimony* from different localities (titration-method). But these experiments do not give the impression of sufficient accuracy, so that it will be best to wait for further determinations, before a final judgement is given.

The investigation described in the above will convince us of the fact, that the higher the degree of accuracy of the method followed and the more vigorous the critical discussion of the deviations eventually met with, the greater the chance to find, that such deviations are of no immediate significance with respect to the problem involved. Independent of their origin, the chemical elements seem really always to have a *constant* ratio of their isotopous components, which ratio is, therefore, *characteristic* for each of them. But then the character of true *constants* of nature must indeed be attributed to the atomic weights of the elements, as ordinarily determined.

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Groningen, Netherlands, April 1924.

¹⁾ J. W. BRÖNSTED und G. VON HEVESY, Zeits. f. anorg. Chem., **124**, 22, (1922).

²⁾ TH. W. RICHARDS, Proceed. Amer. Acad. of Arts and Sc., **23**, 177, (1887); Journ. Amer. Chem. Soc., **24**, 374, (1902).

³⁾ A. D. MONRO, Journ. Chem. Soc. London, **121**, 986, (1922).

⁴⁾ S. D. MUZAFFAR, Journ. Amer. Chem. Soc., **45**, 2009, (1923).

Mathematics. — “*On the place of conformal and projective geometry in the theory of linear displacements*”. By Prof. J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of May 31, 1924).

Introduction. When we try to found differential geometry on the theory of linear displacements, that has developed itself from the idea of pseudoparallel displacement, the difficulty arises, that projective differential geometry (systematically developed by WILCZYNSKI and FUBINI) and conformal differential geometry (systematically developed by CARTAN ¹⁾²⁾ and THOMSEN ³⁾) can not be reduced to a linear displacement. Then CARTAN has enlarged the idea of displacement by adjoining to each point of the manifold, not, as usual, the differential element of this manifold itself, but quite another manifold with a given group of transformations, in the here considered case a projective resp. a conformal group. He then seems to leave principally the linear displacements. This false appearance however only arises because he uses an unusual symbolism, which has not the same invariance as the Ricci-calculus, so that it does not become clear, that in reality we have to do with linear displacements of a more general art, already introduced by R. KÖNIG in 1920 ⁴⁾). These displacements of KÖNIG differ from the ordinary ones only by the number of coordinates in the conjugated manifolds, which is not equal to that, used in the given manifold.

¹⁾ Bull. Soc. Math. 45 p. 57—121, (1917).

²⁾ Ueber konforme Geometrie, Abh. Math. Seminar Hamburg 3 (23) 31—56.

³⁾ a. Sur les espaces généralisés etc., C.R. 174 (22) 734—737.

b. Sur les espaces conformes généralisés etc., C.R. 174 (22) 857—860.

c. Sur la connexion projective des surfaces, C.R. 178 (24) 750—752.

d. Sur les variétés à connexion affine etc., Ann. de l'école norm. sup. 40 (23) 325—412, especially 383 a.f.

e. Les espaces à connexion conforme, Ann de la soc. polon. de math. (23) 171—221.

⁴⁾ Beiträge zu einer allgemeinen Mannigfaltigkeitslehre, Jahresber. d. D. M. V. 28 (20) 213—228.

In the following we will show, that it is perfectly possible to master the general projective and the general conformal manifold of CARTAN (both generalisations of the ordinary manifolds) with the aid of these more general displacements. As this can be done with the same simple invariant equations as are known from the other kinds of geometry and also in the same way, the relations become clearer, and so we can show f. i., that the numbers A_{ikl}^j of CARTAN (*l.c.(e)*) are identical with the components of the quantity of conformal curvature already discovered by WEYL. The far going analogy between conformal and projective geometry becomes very clear, especially in the theorem, proved in the second part, on the relations between affin geometry with invariant geodesic lines (geometry of paths) and projective displacement, a theorem, perfectly analogous to the theorem of CARTAN, demonstrated in the first part, on the relations between conformal RIEMANN geometry and conformal displacement. This second theorem implies of course the quantity of projective curvature, also discovered by WEYL.

I. *The general conformal displacement.*

§ 1. *The euclidean-conformal manifold.* It is known that a hypersphere in R_n ¹⁾ can be given by $n + 2$ homogeneous characteristic numbers $v^j, \alpha, \beta, \gamma, \sigma = b_1, \dots, b_{n+2}$. For $n = 3$ these 5 numbers are pentaspherical coordinates. The quantities v^j and λv^j will be considered algebraically as different, although they correspond geometrically with the same figure. Then there exists a *fundamental tensor* $G_{\alpha\beta}$, such, that the equations

$$(1) \quad G_{\alpha\beta} v^\alpha v^\beta = 0; \quad G_{\alpha\beta} v^\alpha v^\beta = 1; \quad G_{\alpha\beta} v^\alpha w^\beta = 0$$

are characteristic for a *point*, resp. a *unitsphere*, resp. *two mutually orthogonal spheres*. $G_{\alpha\beta}$ may not be confused with the fundamental tensor of the euclidian-metrical geometry in R_n , $g_{\lambda\mu}$, $\alpha, \lambda, \mu, v, \omega = a_1, \dots, a_n$.

When $G_{\alpha\beta}$ is multiplied with a number, nothing is changed in the geometrical interpretation. Such an alteration of $G_{\alpha\beta}$ we do not use however, which does not imply any geometrical restriction. On account of this, there is no difference between covariant and contravariant *quantities*, only between covariant and contravariant *characteristic numbers*, and it is allowed to raise and lower the suffices with the aid of $G_{\alpha\beta}$ and the conjugated tensor $G^{\alpha\beta}$.

¹⁾ We denote with R_n an ordinary n dimensional euclidean-metrical manifold.

point of $\mathbb{E}_n^e$ ¹⁾). Between both $\mathbb{E}_n^e$ in P and a neighbouring point Q there exist up till now no relations whatever. Such a relation is introduced, by indicating, how a quantity belonging to the $\mathbb{E}_n^e$ in P can be displaced to Q . When in every point a normal system u_a^γ is fixed, such that u_0^γ is the point itself, the displacement is defined by the equation:

$$(7) \quad \delta u_a^\gamma = \sum_b A_{a\mu}^b u_b^\gamma dx^\mu \quad \left\{ \begin{array}{l} a, b, c, d = 0, \dots, n+1 \\ \alpha, \beta, \gamma, \delta = 1, \dots, b_{n+2} \\ \kappa, \lambda, \mu, \nu, \omega = 1, \dots, a_n \end{array} \right.$$

or in normal characteristic numbers:

$$(8) \quad \delta u_a^c = \sum_b A_{a\mu}^b u_b^c dx^\mu = A_{a\mu}^c dx^\mu,$$

in which equation the $A_{a\mu}^b$ are $n(n+2)^2$ arbitrary parameters. We now introduce the condition, that $G_{\alpha\beta}$ is invariant, that is, that normal systems always are transformed into normal systems. This leads to the equations:

$$(9) \quad \begin{aligned} A_{0\mu}^0 &= -A_{n+1,\mu}^{n+1} \\ A_{0\mu}^k &= -A_{k\mu}^{n+1}, A_{i\mu}^0 = -A_{n+1,\mu}^i \\ A_{0\mu}^{n+1} &= A_{n+1,\mu}^0 = 0, A_{i\mu}^k = -A_{k\mu}^i, i \neq k \\ A_{i\mu}^i &= 0. \end{aligned}$$

From (9) follows

$$(10) \quad \begin{aligned} \alpha) \quad \delta u_0^\gamma &= (A_{0\mu}^0 u_0^\gamma + \sum_j A_{0\mu}^j u_j^\gamma) dx^\mu \\ \beta) \quad \delta u_i^\gamma &= (A_{i\mu}^0 u^\gamma - A_{0\mu}^i u^\gamma + \sum_j A_{i\mu}^j u_j^\gamma) dx^\mu \\ \gamma) \quad \delta u^\gamma &= (-\sum_j A_{j\mu}^0 u_j^\gamma - A_{0\mu}^0 u^\gamma) dx^\mu. \end{aligned}$$

It may be remarked, that, after (10a) the point P will in general not be transformed into Q . From (8) follows for the covariant differential of an arbitrary quantity of the first degree:

¹⁾ As an example we can take an X_n in R_{n+m} . Then we can consider as $\mathbb{E}_n^e$ in P the $\mathbb{E}_n^e$ of the tangential R_n .

$$(11) \quad \delta v^c = dv^c + \sum_a A_{a\mu}^c v^a dx^\mu$$

$$\delta w_a = dw_a - \sum_c A_{a\mu}^c w_c dx^\mu.$$

The corresponding covariant differential quotients are:

$$(12) \quad \nabla_\mu v^c = \frac{\partial v^c}{\partial x^\mu} + \sum_a A_{a\mu}^c v^a$$

$$\nabla_\mu w_a = \frac{\partial w_a}{\partial x^\mu} - \sum_c A_{a\mu}^c w_c.$$

From (11) and (12) follows indeed that $\delta G_{\alpha\beta}$ and $\nabla_\mu G_{\alpha\beta}$ vanish.

The displacement, obtained in this way, we call *conformal*. The totality of the points of the X_n with the $\mathfrak{E}_n^e$ conjugated to each point and the introduced conformal displacement we call a *general conformal manifold*, $(\mathfrak{E}_n^1)$. A $\mathfrak{E}_n^e$ is a special case of a $\mathfrak{E}_n$. Here to each point is conjugated the $\mathfrak{E}_n^e$ itself, and the displacement becomes a trivial one, each point passes into itself.

In a $\mathfrak{E}_n$ is known in each point what we understand by mutual orthogonal directions in the conjugated $\mathfrak{E}_n^e$. In each $\mathfrak{E}_n^e$ their exist spheres, spheres in neighbouring points can be compared, spheres in not neighbouring points in general not. Transformation of the normal systems has no influence on the displacement, but has of course influence on the parameters $A_{a\mu}^c$. We only consider such transformation of the normal systems, that leave u^γ invariant but for a factor. Such a transformation has the form

$$(13) \quad u^\gamma = \sum_b P_a^b u^\gamma, \quad u^\gamma = \sum_b Q_a^b u^\gamma,$$

in which equation there exists a number of simple relations between the parameters P_a^b , of which we only need the following:

$$(14) \quad \begin{cases} P_0^k = P_0^{n+1} = P_i^{n+1} = 0, & Q_0^k = Q_0^{n+1} = Q_i^{n+1} = 0 \\ P_0^0 Q_0^0 = 1, & P_{n+1}^{n+1} = Q_0^0, & P_0^0 = Q_{n+1}^{n+1} \end{cases}$$

$$(15) \quad \sum_k P_i^k P_j^k = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

¹⁾ An example of a $\mathfrak{E}_n$ can be obtained when we conjugate to each point P of an X_n in R_{n+1} an arbitrary tangential sphere B_n in R_{n+1} . Then a sphere B_{n-1} in the tangential R_n in P is conjugated to the sphere in the tangential R_n in Q , that is cut out of this R_n by a sphere of R_{n+1} , that is orthogonal to B_n and contains B_{n-1} .

From (13) and (14) follows for the transformation of the orthogonal characteristic number:

$$\begin{aligned}
 (16) \quad & 'v_0 = 'v^{n+1} = P_0^0 v_0 \\
 & 'v_i = 'v^i = P_i^0 v_0 + \sum_j P_i^j v_j \\
 & 'v^0 = 'v_{n+1} = P_{n+1}^0 v_0 + \sum_j P_{n+1}^j v_j + Q_0^0 v^0
 \end{aligned}$$

and for the transformation of $A_{0\mu}^0$ and $A_{0\mu}^k$:

$$\begin{aligned}
 (17) \quad & \alpha) 'A_{0\mu}^0 = A_{0\mu}^0 + \frac{\partial}{\partial x^\mu} \log P_0^0 + P_0^0 \sum_j Q_j^0 A_{0\mu}^j \\
 & \beta) 'A_{0\mu}^k = P_0^0 \sum_j Q_j^k A_{0\mu}^j.
 \end{aligned}$$

From (17 ^{β}) and (15) follows, that the n^2 parameters $A_{0\mu}^k$ can be considered as covariant characteristic numbers of n vectors of X_n , which are transformed orthogonally but for a factor with the transformation (13). When we thus write $A_{0\mu}^k = i_{\mu}^k$, and

$$(18) \quad g_{\lambda\mu} = \sum_k i_{\lambda}^k i_{\mu}^k,$$

then the tensor $g_{\lambda\mu}$ of X_n rests invariant with (13) but for a factor:

$$(19) \quad 'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \quad \sigma^2 = P_0^0.$$

When we thus introduce $g_{\lambda\mu}$ as fundamental tensor of the X_n , the conformal displacement of the $\mathfrak{E}_n$ fixes in the corresponding X_n a Riemann geometry but for a conformal transformation. When we write the equation (10 ^{α}) in the form:

$$(10^\alpha) \quad \sigma \frac{u^\gamma}{0} = A_{0\mu}^0 \frac{u^\gamma}{0} dx^\mu + \sum_j \frac{u^\gamma}{0} i_{\mu}^j dx^\mu \frac{u^\gamma}{j},$$

we see, that with a transformation dx^μ with the orthogonal characteristic numbers dx_i corresponds a displacement of $\frac{u^\gamma}{0}$ orthogonal to the system of spheres $\alpha \frac{u^\gamma}{0} + \beta \sum_i \frac{u^\gamma}{i} dx_i$. Hence the directions of X_n in P are conjugated in a one-to-one way to the directions of the $\mathfrak{E}_n^e$ in P , in such a manner, that the orthogonality fixed by $g_{\lambda\mu}$ in X_n corresponds to that fixed by $G_{\alpha\beta}$ in $\mathfrak{E}_n^e$.

From the equation (17 ^{α}) in the form

$$(17^\alpha) \quad 'A_{0\mu}^0 = A_{0\mu}^0 + \frac{\partial}{\partial x^\mu} \log P_0^0 + P_0^0 \sum_j Q_j^0 i_{\mu}^j$$

follows that it is always possible to choose the normal systems in such a way, that $A_{0\mu}^0$ always vanishes. Hence we put

$$(20) \quad A_{0\mu}^0 = 0$$

and only permit such transformations of the normal systems, that leave (20) invariant. The conditions are:

$$(21) \quad \sum_j Q_j^0 \dot{e}_\mu^j = -Q_0^0 \frac{\partial}{\partial x^\mu} \log P_0^0 = \frac{\partial}{\partial x^\mu} Q_0^0,$$

or, when e_i are the orthogonal characteristic numbers of the covariant measuring vectors e_λ of the X_n :

$$(22) \quad Q_i^0 = e_i^\mu \frac{\partial}{\partial x^\mu} Q_0^0.$$

§ 3. *The quantity of curvature of $\mathfrak{E}_n$.* In the X_n itself no displacement whatever is defined. Hence the expression $\nabla_\mu w_\lambda$, in which w_λ is a covariant vector of the X_n , has no meaning. Without defining a displacement in the X_n we can however define already the *alternating* part of $\nabla_\mu w_\lambda$:

$$(23) \quad \nabla_{[\mu} w_{\lambda]} = 1/2 \left(\frac{\partial w_\lambda}{\partial x^\mu} - \frac{\partial w_\mu}{\partial x^\lambda} \right) \quad ^1)$$

from which equation follows:

$$(24) \quad \nabla_{[\omega} \nabla_{\mu} w_{\lambda]} = 0.$$

With the aid of (23) we can now build the second alternated differential quotient $\nabla_{[\omega} \nabla_{\mu]} v^\gamma$ of a quantity v^γ of the $\mathfrak{E}_n$. For, when we write $\nabla_\mu v^\gamma$ as a product of two ideal factors $m_\mu n^\gamma$, m_μ is a quantity of X_n , n^γ a quantity of $\mathfrak{E}_n$, and thus we find in relation to (23) and (11):

$$\begin{aligned} \nabla_{[\omega} \nabla_{\mu]} v^c &= \nabla_{[\omega} m_{\mu]} n^c = n^c \nabla_{[\omega} m_{\mu]} + m_{[\mu} \nabla_{\omega]} n^c = \\ &= 1/2 n^c \left(\frac{\partial m_\mu}{\partial x^\omega} - \frac{\partial m_\omega}{\partial x^\mu} \right) + m_{[\mu} \left(\frac{\partial}{\partial x^\omega} n^c + \sum_b A_{[\omega}^c n^b \right) \\ (25) \quad &= \frac{\partial}{\partial x^{[\omega}} (m_{\mu]} n^c) + \sum_b A_{b[\omega}^c m_{\mu]} n^b \\ &= \frac{\partial}{\partial x^{[\omega}} \left(\frac{\partial}{\partial x^{\mu]} } v^c + \sum_a A_{[\omega}^c v^a \right) + \sum_b \Gamma_{b[\omega}^c \left(\frac{\partial}{\partial x^{\mu]} } v^b + \sum_a A_{[\omega}^b v^a \right) = \\ &= \sum_a \left(\frac{\partial}{\partial x^{[\omega}} A_{\mu]}^c + \sum_b \Gamma_{b[\omega}^c A_{\mu]}^b \right) v^a. \end{aligned}$$

¹⁾ We fix with this equation, that the displacement in the X_n , in which way it may be specified, always is a *symmetrical* one.

When now we write:

$$(26) \quad \mathfrak{E}_{\omega\mu\alpha}^{\dots c} = -2 \frac{\partial}{\partial x^{[\omega}} A_{|\alpha|\mu]}^c - 2 \sum_b A_{b[\omega}^c A_{|\alpha|\mu]}^b,$$

we have

$$(27) \quad 2 \nabla_{[\omega} \nabla_{\mu]} v^c = - \sum_a \mathfrak{E}_{\omega\mu a}^{\dots c} v^a.$$

$\mathfrak{E}_{\omega\mu\alpha}^{\dots c}$ is the *quantity of curvature* of $\mathfrak{E}_n$. When a quantity v^c is moved in the sense of the accepted displacement along the boundary of an X_5 -element $f^{\omega\mu} d\sigma$, the difference between its begin and end-value is

$$(28) \quad D v^c = f^{\omega\mu} d\sigma \sum_a \mathfrak{E}_{\omega\mu a}^{\dots c} v^a.$$

Hence a $\mathfrak{E}_n^e$ is characterised by the property that $\mathfrak{E}_{\omega\mu\alpha}^{\dots c}$ vanishes. From (26) follows for the normal components of $\mathfrak{E}_{\omega\mu\alpha}^{\dots c}$:

$$(29) \quad \begin{aligned} \mathfrak{E}_{\omega\mu 0}^{\dots 0} &= \mathfrak{E}_{\omega\mu 0 n+1} = - \mathfrak{E}_{\omega\mu n+1}^{\dots n+1} = - \mathfrak{E}_{\omega\mu n+1 0} \\ \mathfrak{E}_{\omega\mu 0}^{\dots k} &= \mathfrak{E}_{\omega\mu 0 k} = - \mathfrak{E}_{\omega\mu k}^{\dots n+1} = - \mathfrak{E}_{\omega\mu k 0} \\ \mathfrak{E}_{\omega\mu 0}^{\dots n+1} &= \mathfrak{E}_{\omega\mu 0 0} = \mathfrak{E}_{\omega\mu n+1}^{\dots 0} = \mathfrak{E}_{\omega\mu n+1 n+1} = 0 \\ \mathfrak{E}_{\omega\mu i}^{\dots 0} &= \mathfrak{E}_{\omega\mu i n+1} = - \mathfrak{E}_{\omega\mu n+1}^{\dots i} = - \mathfrak{E}_{\omega\mu n+1 i} \\ \mathfrak{E}_{\omega\mu i}^{\dots j} &= \mathfrak{E}_{\omega\mu i j} = - \mathfrak{E}_{\omega\mu j}^{\dots i} = - \mathfrak{E}_{\omega\mu j i}. \end{aligned}$$

Thus $\mathfrak{E}_{\omega\mu ij}$ is not only alternating in $\omega\mu$, but also in ij :

$$(30) \quad \mathfrak{E}_{\omega\mu ij} = \mathfrak{E}_{[\omega\mu]ij} = \mathfrak{E}_{\omega\mu[ij]}.$$

These identities have the same form as the *first* and *third* identity valuable for the quantity of curvature of a Riemann geometry.

§ 4. *Symmetrical conformal displacements.* u^a is invariant but for a factor. Hence the equation

$$(31) \quad u^a \mathfrak{E}_{\omega\mu\alpha}^{\dots c} = \mathfrak{E}_{\omega\mu 0}^{\dots c} = 0$$

is invariant. From (28) follows, that (31) expresses, that the point P returns in P when P is moved along a closed infinitesimal curve

in the sense of the displacement. A displacement with this property is called *symmetrical*¹⁾. From (31) follows:

$$(32) \quad 1/2 \, \mathfrak{E}_{\omega\mu}^{\dots 0} = - \sum_j i_{[\omega}^j A_{j|\mu]}^0 = 0$$

$$(33) \quad 1/2 \, \mathfrak{E}_{\omega\mu}^{\dots k} = - \nabla_{[\omega}^k i_{\mu]} - \sum_j A_{j[\omega}^k i_{\mu]}^j = 0$$

or, in normal characteristic numbers:

$$(32^a) \quad A_{[ij]}^0 = 0$$

$$(33^a) \quad \nabla_{[m}^k i_{l]} = - A_{[lm]}^k.$$

The remaining characteristic numbers of $\mathfrak{E}_{\omega\mu}^{\dots c}$ are transformed with (13) in the following way:

$$(34) \quad \mathfrak{E}_{\omega\mu i}^{\dots 0} = Q_0^0 \sum_j P_i^j \mathfrak{E}_{\omega\mu j}^{\dots 0} + \sum_{j,k} P_i^j Q_k^0 \mathfrak{E}_{\omega\mu j}^{\dots k}$$

$$(35) \quad \mathfrak{E}_{\omega\mu i}^{\dots k} = \sum_{j,l} P_i^j P_k^l \mathfrak{E}_{\omega\mu j}^{\dots l}.$$

Hence the characteristic numbers $\mathfrak{E}_{\omega\mu i}^{\dots k}$ are transformed into themselves and in such a way, that they can be considered as the characteristic numbers of a quantity of the X_n .

§ 5. BIANCHI's identity. From (27) follows:

$$(36) \quad \begin{aligned} 2 \nabla_{[\xi} \nabla_{\omega} \nabla_{\mu]} v^c &= - \sum_a \nabla_{[\xi} \mathfrak{E}_{\omega\mu]a}^{\dots c} v^a = \\ &= - \sum_a v^a \nabla_{[\xi} \mathfrak{E}_{\omega\mu]a}^{\dots c} - \sum_a (\nabla_{[\xi} v^a) \mathfrak{E}_{\omega\mu]a}^{\dots c}. \end{aligned}$$

But from (27) also follows, with respect to (24)

$$(37) \quad 2 \nabla_{[\xi} \nabla_{\omega} \nabla_{\mu]} v^c = 0 - \sum_a \mathfrak{E}_{[\xi\omega]a}^{\dots c} \nabla_{\mu]} v^a,$$

so that:

$$(38) \quad \nabla_{[\xi} \mathfrak{E}_{\omega\mu]a}^{\dots c} = 0.$$

¹⁾ The mentioned property is for a common linear displacement the criterion for symmetry, th. i. for the vanishing of $\Gamma_{[\lambda\mu]}^{\nu}$. CARTAN uses the expression "sans torsion" that gives rise however to the false supposition that there is a relation with the second curvature of space curves.

This is BIANCHI's *identity* for a conformal displacement¹⁾. From (38) follows:

$$(39) \quad \sum_{b,d} \nabla_{[\xi} \mathfrak{E}_{[\omega\mu]b}^d u_a^b u_d^c = \sum_{b,d} \mathfrak{E}_{[\omega\mu|b]}^d (\nabla_{\xi]} u_a^b) u_d^c + \sum_{b,d} \mathfrak{E}_{[\omega\mu|b]}^d (\nabla_{\xi]} u_d^c) u_a^b = \\ = \sum_b \mathfrak{E}_{[\omega\mu]b}^c A_{a|\xi]}^b - \sum_b \mathfrak{E}_{[\omega\mu|a]}^b A_{b|\xi]}^c.$$

For a symmetrical displacement follows from this equation for $a = 0$:

$$(40) \quad \sum_b \mathfrak{E}_{[\omega\mu|b]}^c i_{\xi]}^b = 0$$

or in orthogonal characteristic numbers:

$$(41) \quad \mathfrak{E}_{[m\bar{k}]}^c = 0,$$

an equation with the form of the second identity valuable for the quantity of curvature of a RIEMANN geometry.

§ 6. On the $\mathfrak{E}_n$ fixed by a RIEMANN geometry given but for conformal transformations. In X_n be given a fundamental tensor but for a scalar factor. Then we can ask to construct a conformal displacement transforming the orthogonal directions with respect to the fundamental tensor always into orthogonal directions. We fix the fundamental tensor in some way and choose n mutually perpendicular unit-vectors. The desired condition is then and only then fulfilled when these vectors can be found but for a factor from the i_{λ}^k of § 2 by an orthogonal transformation. By choosing the coordinate systems and fixing the fundamental tensor in a right way, we can thus always attain that these vectors are identical with the i_{λ}^k . Then for the fundamental tensor the equation (18) holds.

From (33^a) the A_{lm}^k now can be solved. For there is one and only one symmetrical displacement in X_n that leaves $g_{\lambda\mu}$ invariant. For this displacement however holds:

$$(42) \quad \nabla_m i_l^k = - \nabla_m i_k^l;$$

hence (33^a) can with respect to (9) be transformed into:

$$(43) \quad \nabla_m i_l^k + \nabla_l i_k^m = - A_{lm}^k - A_{kl}^m,$$

¹⁾ CARTAN finds equations corresponding to (38), which he calls: "théorème de la conservation de la courbure et de la torsion" i.e. (e) p. 183.

from which equation follows:

$$(44) \quad A_{lm}^k = -\nabla_m i_l^k.$$

The conformal displacement would now be known if also $A_{i\mu}^0$ were known. When we substitute the values obtained in (44) into (26) for $a = i$, $c = k$, we obtain:

$$(45) \quad \begin{aligned} 1/2 \mathfrak{E}_{\omega\mu i}^{\dots k} &= \frac{\partial}{\partial x^{[\omega}} \nabla_{\mu]} i_i^k - \sum_j A_{i[\omega}^j \nabla_{\mu]} i_j^k - i_{[\omega}^k A_{i|\mu]}^0 + i_{[\omega}^i A_{i|\mu]}^0 = \\ &= \nabla_{[\omega} \nabla_{\mu]} i_i^k - i_{[\omega}^k A_{i|\mu]}^0 + i_{[\omega}^i A_{i|\mu]}^0 = 1/2 K_{\omega\mu i}^{\dots k} - 2g[k[\omega A_{i]}^0_{\mu]} \end{aligned}$$

or in orthogonal characteristic numbers:

$$(46) \quad \mathfrak{E}_{mlik} = K_{mlik} + 4g[i[m A_{k]}^0_{l]} \quad ^1),$$

in which K_{mlik} is the quantity of curvature of the RIEMANN geometry belonging to $g_{\lambda\mu}$.

Now A_{kl}^0 can be fixed by the assumption that $\sum_k \mathfrak{E}_{klik}$ vanishes.

Then we find

$$(47) \quad A_{kl}^0 = \frac{1}{n-2} K_{kl} - \frac{1}{2(n-1)(n-2)} K g_{kl}; \quad K_{kl} = \sum_i K_{ikli} \\ K = \sum_i K_{ii},$$

which equation can be solved because $A_{[kl]}^0$ vanishes after (32^a). Then $\mathfrak{E}_{mlik}$ becomes equal to the so-called *quantity of conformal curvature* C_{mlik} , belonging to all RIEMANN geometries that can be obtained by a conformal transformation of $g_{\lambda\mu}$, and hence $\mathfrak{E}_{mlik}$ is a conformal invariant.

Hence we have obtained the theorem:

When in X_n a fundamental tensor is fixed but for a factor, there exists one and only one symmetrical conformal displacement that possesses the orthogonality fixed by this fundamental tensor and for which the equation $\sum_k \mathfrak{E}_{klik} = 0$ holds²⁾.

At the same time we get:

The characteristic numbers $\mathfrak{E}_{mlik}$ are the same as those of the quantity of conformal curvature C_{mlik} of X_n .

II. The general projective displacement.

§ 1. The euclidean-projective manifold. With $\mathfrak{P}_n^e$ we denote an

¹⁾ For $\Lambda_{i\mu}^0 = 0$ the conformal displacement passes into a Riemann one.

²⁾ CARTAN, l.c. (e) p. 184.

euclidean-projective manifold (ordinary "projective space" of n dimensions). A point in $\mathbb{P}_n^e$ can be given by $n+1$ homogeneous contravariant characteristic numbers $v', \alpha, \beta, \gamma, \sigma = b_1, \dots, b_{n+1}$. The quantities $\lambda v'$ and v' will be considered algebraically as different although they represent geometrically the same point. Each point can be written

$$(1) \quad v' = v^\alpha u'_\alpha,$$

in which equations u'_α are the *measuring points*. In the same way every hyperplane can be given by $n+1$ covariant characteristic numbers, w_α . These general characteristic numbers will be immediately changed into others. We choose $n+1$ arbitrary points $u'_\alpha, a, b, c, d = 0, \dots, n$, and call the $n+1$ hyperplanes of their

$(n+1)$ -edron $u'_\alpha; u'_\alpha u'_\alpha = 1$. The characteristic numbers of a point v' or hyperplane w_α with respect to this measuring system are denoted by v^c resp. w_a :

$$(2) \quad v' = \sum_c v^c u'_c, \quad w_\alpha = \sum_a w_a u'_\alpha.$$

We now only allow those measuring systems, that arise from the system u'_α by a linear homogenous transformation with a determinant $\neq 0$. These systems are called *normal systems*.

Hence the quantities

$$(3) \quad \begin{aligned} U^{c_0 \dots c_n} &= u^{[c_0} \dots u^{c_n]} \\ U_{a_0 \dots a_n} &= u_{[a_0} \dots u_{a_n]} \end{aligned}$$

are invariant. This does not contain any geometrical restriction, but the calculations become simpler. The group of transformations of the measuring systems into themselves has the geometrical meaning of the group of projective transformations of the $\mathbb{P}_n^e$.

§ 2. *The general projective manifold.* To each point of an X_n be conjugated a $\mathbb{P}_n^e$, in such a way that P itself is also a point of $\mathbb{P}_n^e$. A relation between the $\mathbb{P}_n^e$ in P and in a neighbouring point Q does not exist. It can be introduced by fixing in each point an

¹⁾ As an example we can take an X_n in $\mathbb{P}_{n+m}^e$. Then we can consider as $\mathbb{P}_n^e$ in P the $\mathbb{P}_n^e$ tangential in P .

arbitrary normal system u^c in such a way that u^c_0 is the point itself.

Then a displacement can be defined by the equations

$$(4) \quad \begin{aligned} \delta u^c_a &= \sum_b A^b_{a\mu} u^c_b dx^\mu = A^c_{a\mu} dx^\mu \\ \delta u^c_a &= - \sum_b A^c_{b\mu} u^b_a dx^\mu = - A^c_{a\mu} dx^\mu, \end{aligned}$$

in which $A^c_{a\mu}$ are $n(n+1)^2$ arbitrary parameters. We will now introduce the condition that normal systems always will pass into normal ones. This condition is

$$(5) \quad \sum_a A^a_{a\mu} = 0.$$

Then the equations for the measuring points are

$$(6) \quad \begin{aligned} \alpha) \quad \delta u^c_0 &= (A^0_{0\mu} u^c_0 + \sum_j A^j_{0\mu} u^c_j) dx^\mu \\ \beta) \quad \delta u^c_i &= (A^0_{i\mu} u^c_0 + \sum_j A^j_{i\mu} u^c_j) dx^\mu \\ i, j, k, l &= 1, \dots, n. \end{aligned}$$

It must be remarked, that the point P after (6 $^\alpha$) does *not* in general pass into Q . From (4) follows for the covariant differential of an arbitrary contravariant or covariant quantity of the first degree:

$$(7) \quad \begin{aligned} \delta v^c &= dv^c + \sum_a A^c_{a\mu} v^a dx^\mu \\ \delta w_a &= dw_a - \sum_c A^c_{a\mu} w_c dx^\mu. \end{aligned}$$

The corresponding covariant differential quotients are

$$(8) \quad \begin{aligned} \nabla_\mu v^c &= \frac{\partial v^c}{\partial x^\mu} + \sum_a \Gamma^c_{a\mu} v^a \\ \nabla_\mu w_a &= \frac{\partial w_a}{\partial x^\mu} - \sum_c \Gamma^c_{a\mu} w_c. \end{aligned}$$

From (3) follows that the differentials of $U^{c_0 \dots c_n}$ and of $V_{a_0 \dots a_n}$ vanish.

The displacement obtained in this way is called a *projective* displacement. The totality of the points of X_n with the $\mathfrak{P}^c_n$ conjugated to each point and the introduced displacement is called a *general*

projective manifold $\mathfrak{P}_n^1$). A $\mathfrak{P}_n^e$ is a particular case of a $\mathfrak{P}_n$. Here to each point is conjugated the $\mathfrak{P}_n^e$ itself and the displacement transforms each point into itself.

Transformation of normal systems has no influence on the displacement, but has influence on the parameters $A_{a\mu}^c$. We only consider such transformations that leave u^γ invariant but for a factor. Such a transformation has the form:

$$(9) \quad {}^c u^c = \sum_a P_a^b u^c \quad ; \quad u^c = \sum_b Q_a^b {}^c u^c,$$

in which there exists between the parameters P_a^b a number of simple relations of which we only need the following:

$$(10) \quad \begin{aligned} \alpha) \quad P_0^k &= Q_0^k = 0 & P_0^0 Q_0^0 &= 1 \\ \beta) \quad P_{[i_1}^{[k_1} \dots P_{i_n]}^{k_n]} &= P_0^0 A_{[i_1}^{[k_1} \dots A_{i_n]}^{k_n]} \\ A_i^k &= \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \end{aligned}$$

From (9) and (10) follows for the transformation of the normal characteristic numbers:

$$(11) \quad \begin{aligned} {}^c v_0 &= P_0^0 v_0 & {}^c w^0 &= Q_0^0 w^0 + \sum_j Q_j^0 w^j \\ {}^c v_i &= P_i^0 v_0 + \sum_j P_i^j v_j & {}^c w^k &= \sum_j Q_j^k w^j \end{aligned}$$

and for the transformation of $A_{0\mu}^c$:

$$(12) \quad \begin{aligned} \alpha) \quad {}^c A_{0\mu}^0 &= A_{0\mu}^0 + \frac{\partial}{\partial x^\mu} \log P_0^0 + P_0^0 \sum_j Q_j^0 A_{0\mu}^j \\ \beta) \quad {}^c A_{0\mu}^k &= P_0^0 \sum_j Q_j^k A_{0\mu}^j. \end{aligned}$$

From (12^a) follows, that the n parameters $A_{0\mu}^k$ can be considered as characteristic numbers of n covariant vectors of X_n , that are transformed linear with the transformation (9), and from (10^a) follows,

¹⁾ As an example we can change an X_n in $\mathfrak{P}_{n+1}^e$ into a $\mathfrak{P}_n^e$ by conjugating to each point P of X_n an arbitrary point C of $\mathfrak{P}_{n+1}^e$ not situated in the tangential $\mathfrak{P}_n^e$. A point in the tangential $\mathfrak{P}_n^e$ in P is then conjugated to its central projection from C on the tangential $\mathfrak{P}_n^e$ in a neighbouring point Q . CARTAN l.c. (c).

that for $P_0^0 = 1$ the determinant of the transformation is $+1$. For these n vectors and their reciprocals we write $a_\mu^k a_\gamma^i$:

$$(13) \quad a_\mu^k = \Gamma_{0\mu}^k \quad ; \quad a_\gamma^i a_\gamma^k = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Then follows from (6^a) that with a displacement dx^μ with the characteristic numbers dx^j with respect to the measuring system a^γ corresponds a displacement of u^c in the direction of the point j $\sum_j u^c dx^j$. Hence the directions of X_n in P are conjugated in a one-

to-one way to those of the $\mathfrak{P}_n^e$ in P . From the equations (12^a) follows that it is always possible to choose the normal systems in such a way that $A_{0\mu}^0$ everywhere vanishes. We thus assume $A_{0\mu}^0$ to vanish and only permit such transformations of normal systems that leave $A_{0\mu}^0$ zero, th. i. for which the equations hold:

$$(14) \quad \sum_j Q_j^0 a_\mu^j = -Q_0^0 \frac{\partial}{\partial x^\mu} \log P_0^0 = \frac{\partial}{\partial x^\mu} Q_0^0.$$

§ 3. *The quantity of curvature of $\mathfrak{P}_n$.* As in § 3 we assume from the displacement in X_n only that it is a symmetrical one. Then the equations (I 23) and (I 24) hold and in the same way as in I § 3 we find:

$$(15) \quad 2 \nabla_{[\omega} \nabla_{\mu]} v^c = -\mathfrak{P}_{\omega\mu a}^{\dots c} v^a,$$

in which equation

$$(16) \quad \mathfrak{P}_{\omega\mu a}^{\dots c} = -2 \frac{\partial}{\partial x^{[\omega}} A_{|\mu|]}^c - 2 A_{b[\omega}^c A_{|\mu|]}^b.$$

$\mathfrak{P}_{\omega\mu a}^{\dots c}$ is the quantity of curvature of $\mathfrak{P}_n$. A $\mathfrak{P}_n^e$ is characterized by the vanishing of this quantity.

From (5) and $A_{0\mu}^0 = 0$ follows:

$$(17) \quad \begin{aligned} \sum_a \mathfrak{P}_{\omega\mu a}^{\dots a} &= - \sum_a \Gamma_{0[\omega}^a \Gamma_{|\mu|]}^0 - \sum_{a,j} \Gamma_{j[\omega}^a \Gamma_{|\mu|]}^j \\ &= - \sum_j \Gamma_{0[\omega}^j \Gamma_{|\mu|]}^0 - \sum_j \Gamma_{j[\omega}^0 \Gamma_{|\mu|]}^j = 0, \end{aligned}$$

an equation, which has the form of the condition of conservation of volume in an affine displacement.

§ 4. *Symmetrical projective displacements.* u^c is invariant but for a factor. Hence the equation

$$(18) \quad u^a \mathfrak{P}_{\omega\mu a}^{\dots c} = \mathfrak{P}_{\omega\mu 0}^{\dots c} = 0$$

is invariant. From (I 28), which equation holds also for $\mathfrak{P}_{\omega\mu a}^{\dots c}$, follows, that (18) expresses that the point P returns in P when P is moved along a closed infinitesimal curve in the sense of the displacement. A displacement with this property is called *symmetrical*.

From (18) follows:

$$(19) \quad \frac{1}{2} \mathfrak{P}_{\omega\mu 0}^{\dots 0} = - \sum_j^j a_{[\omega} A_{j|\mu]}^0 = 0$$

$$(20) \quad \frac{1}{2} \mathfrak{P}_{\omega\mu 0}^{\dots k} = - \nabla_{[\omega}^k a_{\mu]} - \sum_j^j A_{j[\omega}^k a_{\mu]}^j = 0,$$

or, in normal characteristic numbers:

$$(19^a) \quad A_{[ij]}^0 = 0$$

$$(20^a) \quad \nabla_{[m}^k a_{l]} = A_{[lm]}^k.$$

The remaining characteristic numbers are transformed in the following way:

$$(21) \quad \mathfrak{P}_{\omega\mu i}^{\dots 0} = Q_0^0 \sum_j^j P_i^j \mathfrak{P}_{\omega\mu j}^{\dots 0} + \sum_{j,k}^j P_i^j Q_k^0 \mathfrak{P}_{\omega\mu j}^{\dots k}$$

$$(22) \quad \mathfrak{P}_{\omega\mu i}^{\dots k} = \sum_{j,l}^j P_i^j P_l^k \mathfrak{P}_{\omega\mu j}^{\dots l}.$$

Hence the characteristic numbers $\mathfrak{P}_{\omega\mu i}^{\dots k}$ are transformed into themselves and in such way, that they can be considered as the characteristic numbers of a quantity of X_n .

§ 5. *BIANCHI's identity*. In the same way as in I § 5 we get:

$$(23) \quad \nabla_{[\xi} \mathfrak{P}_{\omega\mu]a}^{\dots c} = 0.$$

This is *BIANCHI's identity* for a projective displacement. For a symmetrical displacement we can deduce from this equation in the same way as in I § 5:

$$(24) \quad \mathfrak{P}_{[m]i}^{\dots c} = 0.$$

For a symmetrical displacement can be found from (17) and (24):

$$(25) \quad \sum_i \mathfrak{P}_i^i [kl] = 0.$$

§ 6. On the $\mathfrak{P}_n$ that is fixed by the geodesic lines of an affine displacement. We can now ask when a symmetrical projective displacement determines in the X_n the same geodesic lines as a given symmetrical affine displacement. The affine displacement be given with respect to the vectors a^k_i :

$$(26) \quad \nabla_\mu a^k_i = \sum_j \Gamma_{i\mu}^j a^k_j = \Gamma_{i\mu}^k.$$

It can easily be proved that the projective displacement has then and only then the same geodesic lines as the affine displacement when

$$(27) \quad A_{i\mu}^k = \Gamma_{i\mu}^k + p_i A_\mu^k + p_\mu A_i^k,$$

in which p_λ is an arbitrary vector. Of all symmetrical affine displacements with the same geodesic lines only one has the properties of conservation of volume and of leaving invariant just the quantity $a^{[v_1} \dots a^{v_n]}$. If we assume that $\Gamma_{i\mu}^k$ are the parameters of this displacement, then $\sum_i \Gamma_{i\mu}^i = 0$ and from (5) and (27) follows $p_\lambda = 0$ or:

$$(28) \quad A_{i\mu}^k = \Gamma_{i\mu}^k = \nabla_\mu a^k_i.$$

Substituting the value obtained in (28) into (16) for $a = i, c = k$, we find:

$$(29) \quad \mathfrak{P}_{\omega\mu i}^{\dots k} = R_{\omega\mu i}^{\dots k} - 2 a_{[\omega}^k A_{|\mu|}^0],$$

or, in normal characteristic numbers:

$$(30) \quad \mathfrak{P}_{mli}^{\dots k} = R_{mli}^{\dots k} - 2 A_{[m}^k A_{|i|l]}^0)^1,$$

in which $R_{mli}^{\dots k}$ is the quantity of curvature of the affine displacement determined by $\Gamma_{\mu i}^k$. Now A_{il}^0 can be fixed by the condition that $\sum_k \mathfrak{P}_{kli}^{\dots k}$ vanishes. Then we find

$$(31) \quad A_{il}^0 = \frac{1}{n-1} R_{il}, \quad R_{il} = \sum_k R_{kil}^{\dots k},$$

which equations can be fulfilled because of the vanishing of $A_{[il]}^0$ and $R_{[il]}$ after (19^a) and because of the conservation of volume of the affine displacement. Then $\mathfrak{P}_{mli}^{\dots k}$ becomes identical with the so-called

¹⁾ For $A_{il}^0 = 0$ the projective displacement passes into an affine one.

quantity of projective curvature $P_{mli}^{\dots k}$ belonging to all affine displacements with the same geodesic lines, and hence $\mathfrak{P}_{mli}^{\dots k}$ is invariant with all transformations of the affine displacement, that leave geodesic lines invariant.

We thus have obtained the theorem:

When in an X_n of an affine displacement only the geodesic lines are given, there exists one and only one symmetrical projective displacement, whose geodesic lines are identical with those given lines and for which the equation $\sum_i \mathfrak{P}_{kii}^{\dots k} = 0$ holds ¹⁾.

At the same time we have found:

The characteristic numbers $\mathfrak{P}_{mli}^{\dots k}$ are the same as those of the quantity of projective curvature $P_{mli}^{\dots k}$ of X_n .

¹⁾ During the correction of the proofs of this communication Mr. CARTAN kindly send me proof of his new paper "Sur les variétés à connexion affine", Bull. Soc. Math. 1924. In this paper Mr. CARTAN has also given a proof of the here proved theorem.

Physiology. — "*Emanation and Calcium*". By H. ZWAARDEMAKER, T. P. FEENSTRA and M. E. J. M. STEYNS.

(Communicated at the meeting of March 29, 1924).

The well-known relationship which must exist in physiological systems between potassium and calcium induced the authors to examine whether such a relationship could also be discovered between emanation, the most ideal potassium-substitute¹⁾, and calcium. A few preparatory examinations had already shown that the ordinary formula K/Ca could certainly not be applied here without alteration.²⁾

When one of us (S) had invented a contrivance by means of which it was possible to keep an emanation solution shut off from the air, and yet to use it as a perfusing fluid with only a scarcely noticeable change in hydrostatic pressure, the last obstacle was removed to a more systematic research. This contrivance consists of a small Mariotte bottle filled to the brim, the tube for the air supply being covered with a limp rubber bag (condome). When the fluid escapes from the lower opening the bottle can fill itself with air from above; which however remains entirely separated from the fluid.

The experiments were done with spontaneously pulsating hearts of frogs and lampreys (*Petromyzon fluviatilis*). In the first a Kro-necker cannula was introduced into the ventricle, in the second a simple glass tube was placed in the inferior vena cava, via the hepatic vein. In the first case therefore only the ventricle was perfused, in the second the whole heart. Registration was effected by suspending the ventricle, respectively ventricle and atrium.

The general course adopted was the following: First the organ pulsated during a quarter of an hour with a suitable potassium-Ringer solution. If the circumstances made it desirable, the Ca-ion also was partially or wholly removed from the fluid which was intended to produce a standstill. And lastly when it stood still, the organ was perfused with the fluid we wished to examine.

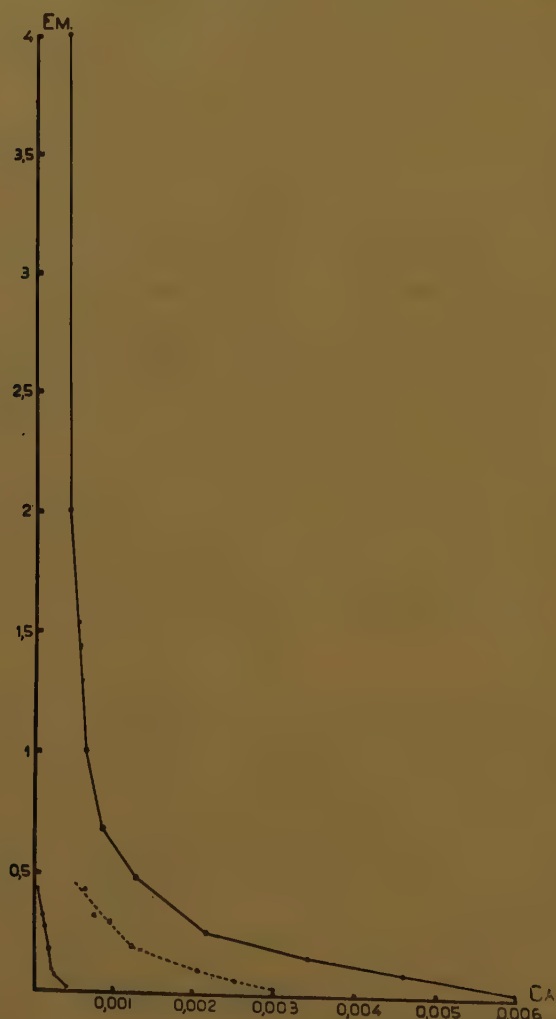
The first series of experiments determined the smallest quantity

¹⁾ H. ZWAARDEMAKER, Verslag Kon. Akad. v. Wetensch. Amsterdam, Deel XXVI (1917), p. 776; these Proc. Vol. XX, p. 773.

²⁾ H. ZWAARDEMAKER et T. P. FEENSTRA, C. R. Soc. de Biol. T. 84, p. 377, 1921.

of calcium ion and the smallest quantity of emanation, which, buffered with bicarbonate of sodium, (100 mgr. p. L.) and mixed with a solution of sodium chloride (6.5 gr. p. L.) just admitted of a regular pulsation.

A second series of experiments determined the largest quantity of calcium ions and the greatest quantity of emanation, which,



buffered with bicarbonate of sodium (200 mgr. p. L.) and added to a sodium chloride solution (6.5 gr. p. L.) just admits of or prevents a regular pulsation.

As a starting point for the addition of calcium chloride we used a concentrated aqueous solution with a specific weight of 1.2294 and containing 25 % dry CaCl_2 (according to the tables of LANDOLT and BÖRNSTEIN 4th edit. p. 470).

As a starting-point for the addition of emanation we used 1 microcurie, which we acquired by leaving a solution of radium (1 to a million) in bottles of 100 cm³. to itself during a month, after which the emanation was distilled off into a vacuum. The emanation gas obtained in this manner was then added to the physiological sodium chloride solution to be used in the various experiments.

In the foregoing graph we have summarized the results obtained. On the axis of the abscissae the different quantities of calcium in gram molecules per L. have been noted, on the axis of the ordinates the quantities of emanation in microcuries p. L. Perhaps the last-named figures are slightly too high, as a certain loss of emanation ensues from a small dead space in the outlet tube of the emanation bottle.

The highest line is the line of maximum quantities. Above this line the heart stands still, and below it it pulsates. The lowest line is the line of minimum quantities. Below the line the heart stands still and above it it pulsates. In cases responding to points above the maximum line no electrocardiogram was visible; below the minimum line only when sufficient emanation was present, f.i. 1/100 microcuries p. Litre.

The zone between these two curves is the zone of regularly pulsating hearts. Such an automatism can be maintained for any length of time if only care is taken to fix up a new Mariotte bottle when the old one has been emptied.

It is of course possible to discover the horizontal centres of these two lines. These points answer to the following.

Emanation in curies	Calcium in gramme molecules	Em. $\times$ Ca.
0.5×10^{-7}	5×10^{-3}	2.5×10^{-10}
0.8×10^{-7}	3.6×10^{-3}	2.9×10^{-10}
1×10^{-7}	2×10^{-3}	2.0×10^{-10}
1.5×10^{-7}	1.5×10^{-3}	2.25×10^{-10}
2×10^{-7}	1.3×10^{-3}	2.6×10^{-10}
3×10^{-7}	0.82×10^{-3}	2.46×10^{-10}
4×10^{-7}	0.75×10^{-3}	3.0×10^{-10}

As may readily be seen a line drawn through the above-mentioned points, (see dotted line), satisfies the equation:

$$\text{Em} \times \text{Ca} = \pm \text{constant.}$$

We have not dared to presume upon the biological significance of this relationship. One of us (F) formerly found: $\frac{K}{Ca} = \pm 2$ (K and Ca in gram molecules) using the same object to experiment upon; we now found: $Em \times Ca = 2.5 \times 10^{-10}$ (averaged), Em in curies, Ca in gram molecules.

Both formulæ point to a relationship between the radioactive element and calcium. A closer examination of the first relationship has proved that it is an exceptional case of a general balance of ions, which always finds expression in a quotient. Such a quotient is supposed by some authors to mean an antagonism; SPIRO¹⁾ has, however shown that one should speak of pseudo-antagonism, as the quotient only expresses that numerator and denominator weaken, but by no means neutralize one another.

In the case before us we are dealing with neither antagonism nor pseudo-antagonism, for in the product $Em \times Ca$ both factors support one another. A closer examination shows that both factors have a limit, which lies at approximately $\frac{1}{1000}$ microcuries per Litre for emanation, and at a few milligram per L. for Ca, and in the latter case the electrocardiogram remains visible if only sufficient microcuries are present. Calcium can, therefore be dispensed with, not emanation. In other words, calcium is not an emanation substitute, but an emanation-intensifier.

We have not yet been able to discover a physical cause for the intensifying influence of Ca on emanation for the automatisms we examined. Emanation is a pure alpha radiator and a heart pulsating on emanation entirely possesses the character of an organ in alpha automatism, with respect to paradoxes and radiophysiological equilibria. The intensifying influence which Ca exerts must, therefore, be an influence on the alpha radiators, unless one may accept that the presence of Ca aids the adsorption of emanation. For emanation is highly absorbed not only by charcoal²⁾ but also by animal tissues. As soon as occasion offers we will try to discover whether Ca exercises any influence on this adsorption. If this were the case it would offer an explanation, for it is not the emanation atoms in the centre of the perfusing fluid which are of biological significance. It is the atoms adsorbed by the cells which revive the automatism which was forfeited by removing the radioactive element. In order

¹⁾ K. SPIRO, Baseler Antrittsvorl. 1921, and Schweiz. med. Wochenschr. 1921, No. 20.

²⁾ Mixtures are of great importance. Cocoa-nut absorbs best, compare St. MEYER v. SCHWEIDLER, Radioaktivität 1916, p. 329.

to obtain a similar result by radiation from without the bombardment of alphaparticles must of necessity be a million times stronger.

Some time ago one of us (Z) examined this, together with GRJNS ¹⁾, in experiments in which a pulsating heart was brought to a standstill by removing the potassium after which the regular pulsations were revived by radiation with polonium, applied externally.

The experiments took place in completely quiet surroundings (camera silentia of the laboratory) so that no jerks or shocks could reach the heart. Nevertheless the hearts revived spontaneously, while the perfusion with potassium-free Ringer continued. This research was repeated a short time ago by VOORSTAD ²⁾ on a more systematic and extensive scale, with the same results. The number of particles emitted in these experiments is almost infinitely large, as appears from the strong scintillating effect which polonium produces; only a few of these, however, penetrate to the depths as their penetrating power is fairly limited. As far as we could discover the Ca present in the fluid had neither a stimulating nor an inhibitory influence. In the case we speak of in this communication we substituted the microradioactivity of a perfusing fluid for the macroradioactivity of the external radiation.

We are of opinion that the fact that the lost automatism can, all the same, be revived with the same certainty, must be ascribed: 1° to the large area of the walls of the lacunes in the muscles, 2° to the greatly increased concentration, which it attained by the adsorption on the limiting membrane between cells and perfusing fluid, 3° the almost entire lack of distance between the source of radiation and the seat of automatism.

¹⁾ ZWAARDEMAKER en GRJNS, Arch. néerl. t. 2, p. 500, 1918. Comp. also Verslag Kon. Akad. v. Wetensch. Amsterdam Dl. XXVIII (1919), p. 370; these Proc. Vol. XXII, p. 383.

²⁾ J. N. VOORSTAD. Diss. Utrecht 1923.

Zoölogy and Palæontology. — “*On the Brain Quantity of Specialized Genera of Mammals*”. By Prof. EUG. DUBOIS.

(Communicated at the meeting of May 3, 1924).

It may be considered as well established now that from small to large mammal species of *equal organisation* the weight (or the volume) of the brain, E , increases as the $5/9$ power of the body weight, P . Thus from the Cat to the Tiger, from the Mouse to the Rat, from the Roe to the Red Deer, with body weights which are to each other as 1: 55, 1: 17, and 1: 12. I have shown in some earlier communications that the same law holds for all classes of vertebrata, that the relation found rests on equal increase of the separate volume and the number of the nerve elements (neurones), both proportional as $P^{5/18}$ ¹⁾, and furnished proofs that render the rational character of the law plausible.

With *different organisation* the weight (or the volume) of the brain depends, besides on the bulk of the body, also on another factor, indicating the degree of the organisation of the brain, the coefficient of cephalisation α . General is therefore $E = \alpha P^{5/9}$, and the particular α characteristic of each group of species of equal organisation

$$= \frac{E}{P^{5/9}}.$$

The comparative value of this coefficient to be calculated thus is in very good harmony, over the whole of the orders, with the rank assigned to each of them in the zoölogical system, on the ground of what we know of its degree of organisation, and also on phylogenetic considerations.

An exception makes the high α in the order of the Cetacea, and between many closely allied families and genera of all orders there

¹⁾ Lately this found an unintended confirmation through an investigation of L. LAPICQUE and A. GIROUD. According to their counting of the nerve fibres in the 12th pair of ventral and dorsal spinal cord roots of *Mus norvegicus* and *Mus musculus* the total amounted to 819 for the large, and 393 for the small muridae species (both values the mean of 9 indiv.). With the ratio of the body weights of the species, 17:1, this gives the relation-exponent 0.26, which is very near $5/18$. (Comptes rendus de la Société de Biologie. Tome 89, p. 938—939. Paris 1923).

exist as considerable differences in the value of α as between these largest groups of mammals.

We consider it a matter of course — because in these matters we are still used to think anthropocentrically — that Man is particularly distinguished in this respect. But just as greatly as Man differs from his nearest relations, the Elephant differs from his. This drew at once the attention, when the law in question became known. It soon appeared that there are many of these “deviations”.

In order to obtain a better insight into the significance of the great differences between the oldest Tertiary and later allied mammal genera and of the quantity of the brain of Pithecanthropus, large in comparison with the Manlike Apes, small in comparison with Man, I have, of late years, made a closer inquiry into the deviations in the value of α in allied existing mammal genera. I have in particular determined the capacities of a number of mammalian crania, mostly from the *Rijks Museum van Natuurlijke Historie* at Leiden, and I have collected, and as much as possible verified, all that is to be found in the literature about determinations of brain quantities (weights or volumes) and body weights¹⁾.

The cranial capacity is often a better measure of the brain quantity of a mammal species than the brain quantity directly determined. This can be calculated with sufficient accuracy from the capacity of the cranium, and does not appreciably differ from the former in small skulls. For few weighings of brains have been made on animals in the natural state, most on animals that have died of disease in zoölogical gardens, and then the individuals had not even always reached their full growth; in which case their body weights remain much, not seldom very much, below that characteristic of the species. In many cases, especially in monkeys, the body weight determined thus may be only 50 %, and less of the natural weight of the species. It is certainly regrettable that of most mammalian species (not to mention the other classes of Vertebrata) little is known of so principal a character as the natural body weight. Important questions of the nature of the subject of this communication are, for this reason, liable to investigation only in a limited degree, and without the

¹⁾ Some body weights were determined by myself. I am indebted for data about body weights of American Apes in the natural state to Prof. A. HRDLÍČKA and to my son J. DUBOIS, for data about rodents and carnivora weighed living in the zoölogical gardens at Amsterdam to Mr. A. F. J. PORTIELJE.

Some body weights had to be estimated by comparison with homomorphic other species, which could be done without errors of importance as appears from other checked estimations.

greatest circumspection use of the available data of body weights may easily lead to wrong conclusions¹⁾. To mention a single example, what can be expected from conclusions resting on the assumption of a body weight of 12,6 kg., from zoological garden specimens, for the adult Seal (*Phoca vitulina*), while the natural weight of the species amounts to about *five* times as much! Of species living in the natural state only the individual maximum weights are often given. Many shoals must be avoided before we arrive at a correct idea about the relation between brain weight and body weight. With all these difficulties I have only been able to establish the comparative value of the coefficient of cephalisation α of a limited number of closely allied genera and families. In this I also restricted myself to the Placental Mammalia. Nevertheless a uniformity of the relations was thus found.

The most important result of the investigation is that the deviations are *always great* and consist in *doubling of the whole or of a very great part (half) of the brain quantity calculated for equal body weight*. These quantities are, therefore, mostly to each other as 1:2:4, but sometimes as 1:1½:3:6. In most cases doubling of the cerebrum and the cerebellum takes certainly place, whereas some of the other brain parts, of small volume, perhaps do not participate in the increase. That in most cases the increase remains *slightly* below the relations mentioned is probably to be accounted for in this way.

As regards *fossil* Mammals there were too few accurate quantitative data available. Probably the volume of the cerebral hemispheres of the old-eocene *Arctocyon* was only 1/3 of that of an existing domestic dog of equal bulk. The eocene *Moeritherium* was on a par with the existing *Procavia*²⁾.

In all cases the increase of the α of existing mammalian genera appeared to go hand in hand with, and was, therefore, no doubt owing to *specialisation of the organisation of the animal organic systems*, i.e. on the increase of the outer world and of the inner world (VON UEXKÜLL's "Umwelt" and "Innenwelt") of the animal.

In the order of the existing Insectivora the Soricidae reach scarcely more than 1/60 of the α of Man. But a relation, *Talpa*, has specialized to get its food by burrowing "runs" with its large spade-like hands. A relation of the Centetidae, which are at the same level as the Soricidae, *Potamogale*, the Otter-Shrew, is through its oar-tail,

¹⁾ Some of my earlier calculations required correction on this account.

²⁾ I am indebted to Dr. A. SMITH WOODWARD of the British Museum (Natural History) of London for an endocranial cast of *Moeritherium*.

an admirably rapid fish-catching swimmer¹⁾. *Galeopithecus*, which has adopted a diet of tree-leaves, and steers itself through the air from one tree to another at a considerable distance by means of its parachutelike expansion of the skin, has like *Talpa* and *Potamogale*, reached double the κ of that of the families at the lowest stage of the Insectivora. In the family of *Talpidae* *Myogale moschata* has specialized to an excellent swimmer through the organisation of its feet and tail, and thus obtained the same κ as *Talpa*. As in *Talpa*, its proboscis too will certainly have contributed to this. In the Squirrel-Shrew, *Tupaia*, which is provided with a steering tail (uropere) and which uses its fore-paws as hands (podochire), besides being optically better organized, the κ rises to the fourfold of this lowest Insectivora level.

In the Microchiroptera the κ is at $1\frac{1}{2}$ times that level. The frugivorous Megachiroptera are distinguished from the Insectivorous Microchiroptera by this that their feet are used as hands, with which they can hold fruit and catch fish, and by their excellent optical sense. They are also much more agile fliers. Their κ is found to be trebled.

The Pine-Marten, *Mustela martes*, has specialized as nimble hunter of squirrels, and like the equally agile Beech-Marten, *Mustela foina*, it has obtained double the value of κ of its nearest relations of the genus *Putorius*: the Polecat, the Stoat, and the Weasel, and also of the Wolverine, *Gulo*; likewise the Otter, *Lutra*, as skilful swimmer in the family of the Mustelidae. The still more specialized Sea-Otter, *Lutax*, has again a κ $1\frac{1}{2}$ times that of the common Otter genus. The Bears of the genus *Ursus*, through their power of climbing and the use they make of their fore-paws as hands, have reached double the value of the κ of their lowest existing relations, the Viverridae. The Felidae exceed the Viverridae only $1\frac{1}{2}$ times. The best and quickest climber in the Bear family, the Malay Bear, *Helarctos*, has $1\frac{1}{2}$ times the κ of the genus *Ursus*, and is in this respect the equal of the Anthropoid Apes. The fin-footed Carnivores, Pinnipedia, though primitive in many respects, are on a line with *Ursus*, through their more perfect specialisation for aquatic life than *Lutra*.

The most perfectly specialized swimmers among the Mammalia are the Delphinoceti, which have become fish-like in their general outward form. With the Elephant and some American Monkeys they

¹⁾ I owe the determination of cranial capacities of the latter Insectivore, which is as rare in the musea as it is interesting, to Mr. M. A. C. HINTON and W. P. PYCRAFT of the British Museum.

come nearest to Man as regards the value of their κ . The *Mystacoceti*, which are much less good swimmers than the *Delphinoceti*, have only $\frac{1}{3}$ of the κ of the latter.

Among the *Edentata* *Myrmecophaga* has a κ $1\frac{1}{2}$ times as great as *Bradypus*. The Ant-eater is in this respect the equal of *Orycteropus*, with which it can be well compared in the mutual size of the brain divisions. Though nothing is known of the nearer relationship of *Manis* with any of the other *Edentata*, it may be stated here that in this genus the κ has only $\frac{1}{3}$ of the value of that in *Orycteropus*.

The Hares, *Leporidae*, all being swift of foot, and the Jumping Mice, *Dipodoidea*, living in steppes and deserts, have as fast runners twice the κ of the Mice, *Myoidea*; the Squirrels, of the genus *Sciurus*, as agile uroptere and podochire Rodents, the threefold κ . As fast runner also the Patagonian Mara, *Dolichotis*, in its habits resembling a hare, reaches $1\frac{1}{2}$ times the κ of its nearest relations, the (wild) *Cavia* species. The same applies to the swimming specialist of the family, the *Capibara*, *Hydrochoerus*.

The most primitive Ungulate existing, the stealthily moving *Tragulus*, has only half the κ of its locomotively highly specialized modern relations, the *Cervicornia* and the *Cavicornia*.

The eocene probable ancestor of *Elephas*, *Moeritherium*, possessed only $\frac{1}{6}$ of the κ of its modern descendant, which has developed a very perfect feeler and prehensile apparatus with finger in its proboscis, through which *Elephas* has risen above the Manlike Apes. The *Sirenia*, which also descend from *Moeritherium*, have obtained $1\frac{1}{2}$ the latter's κ through their organisation for aquatic life. The existing primitive relation of the Elephant, *Hyrax* or *Procavia*, has like *Moeritherium*, $\frac{1}{6}$ of the latter's κ . *Rhinoceros* has double the κ of *Procavia*. Of the closely allied genera *Hippopotamus* and *Sus*, the latter, higher in motive respect, possesses double the κ of the former genus. *Tapirus* is equal with *Sus*.

The genus *Lemur* of Madagascar possesses the sixfold κ of the primitive Insectivorous genera. The Aye-Aye, *Daubentonia*, of this island, reaches double the κ of *Lemur*, through the very remarkable specialisation of its fingers, particularly of the extremely slender and "ghostly" middle finger, which serves as probe and "sharp spoon". It may be said in passing that the κ of *Tarsius* is at most equal to that of *Nycticebus*, perhaps only $\frac{2}{3}$ of it.

Among the American monkey genera the κ rises from *Callitrix*, which in its stage of organisation bears rather a squirrel-like than a simian character, to the genuinely simian *Saimiri*, and from this to *Cebus*,

which not only possesses a prehensile tail, in contrast with the half limp tail of Saimiri, but particularly a so highly developed hand-organisation, that in the natural state, it makes use of stones to break open nuts, in the ratio of 1:2:4. Between Alouata and Cebus the ratio is 1:3. On the same line with Cebus is Ateles, which possesses, indeed, a less perfect hand than Cebus, in so far as it lacks the thumb, but which, to make up for this, possesses in its very perfect prehensile tail a fifth hand, a tactile feeler and an excellent instrument, comparable with the proboscis of the Elephant. Both genera have risen above the Manlike Apes with a $1\frac{1}{2}$ times greater κ . But in Cebus and Ateles the evolution of the Anthro-poidea in America has reached its culminating point. The Anthro-poidea of the Old World culminate in Man, through their more consistent and absolute hand specialisation attended with the loss of the tail (besides higher organisation of the visual sense), in consequence of the division of labour of the pairs of limbs, which has already proceeded to temporarily semi-erect locomotion in the Manlike Apes. In the latter, the Simiidae, κ , $\frac{1}{4}$ of that of Man, is $1\frac{1}{2}$ times as great as in the Cynopithecidae. Hence Man has the same ratio 6:1 to the latter as Elephas to Proavia. Pithecanthropus has half the κ of Man, and double that of the Manlike Apes¹⁾.

It is evidently the increase of the number of the nerve elements, of the neurones, which in all these cases increases the quantity of the brain. For the determinations of the cell-density in the gray cortex of the brain of Apes made for an entirely different purpose by OTTO MAYER²⁾ teach that the absolute number of the cells increases in direct ratio to the κ of the genera.

As was already said above, in Callithrix, Saimiri, and Cebus the

1) If from the close resemblance of the femur of Pithecanthropus with the human femur the conclusion is erroneously drawn that the organisation of the upper pair of limbs was also the same, κ must be put at $\frac{2}{3}$ of that of Man, when the body weight is estimated in accordance with human proportions. From the peculiarities of the fossil skull it can, however, be derived that the erect attitude had not yet reached human perfection, and that in consequence of this the upper pair of extremities, and with them the upper part of the body, were comparatively heavier than with human proportions. With double the value of κ of the Manlike Apes and half that of Man, Pithecanthropus must have weighed about 100 kg. This is a very plausible body weight.

2) OTTO MAYER, Mikrometrische Untersuchungen über die Zelldichtigkeit der Grosshirnrinde bei den Affen. Journal für Psychologie und Neurologie. Bd. 19, p. 233—251. Leipzig 1912. From Tabelle I, p. 237, I calculate over the whole of the cortex of the cerebrum on an average per 1 mm². 3448 cells in Callithrix, 3603 cells in Saimiri, and 3581 cells in Cebus.

values of x are to each other as 1: 2: 4, the cell-density being found the same.

The doubling of this coefficient is, accordingly, the consequence of a division of cells all over the brain. If the value becomes $1\frac{1}{2}$, times greater, the cell-division refers only to half of the whole quantity of the brain (all the same a very great part), while in the other half the cells remain undivided. The three-fold x means two cell-divisions, one of which was general and one partial. Hence in all cases of specialized animal organisation we do not see only limited parts of the brain, localized regions, increase, but the whole or a very large part ¹⁾).

From the cell-countings in the different convolutions of the cerebral cortex of Man by HAMMARBERG ²⁾ HELEN BRADFORD THOMPSON ³⁾ calculated a total of 9200 million cells. The number of 8590 million lying close to it corresponds to 33 cell-divisions, from the germ-cell to shortly before birth, when the brain cells cease dividing. In the Manlike Apes only 31 cell-cleavages then take place; in *Pithecanthropus* this number must have been 32. One segmentation more or less determines the degree of development of the central organ of the animal life, the extent of the outer and the inner world of the animal. The brain quantity being regulated in ratio to the whole body, the specializing cell-division must refer to the first differentiation as brain cells; the *first* cell of the whole organ or of a very large part of the organ is doubled. This development resting on a single cell-division takes necessarily place in leaps, "per saltum", not continuously. And now it is also possible to understand something of the great leaps of the brain development in the specialisation of the essentially animal organisation in the world of Mammals after the beginning of the Tertiary era, and in the specialisation of the essentially animal organisation in the world of Reptiles after the Mesozoic era.

Besides the animal organisation proper, of the nervous system

¹⁾ Already in 1902 G. ELLIOT SMITH emphatically pointed out that the extension which the neopalium has obtained in the human brain is a general one, not restricted to "any localized areas". (Catalogue of the Physiological Series of Comparative Anatomy Contained in the Museum of the Royal College of Surgeons, p. 466. London 1902).

²⁾ CARL HAMMARBERG, Studien über Klinik und Pathologie der Idiotie, nebst Untersuchungen über die normale Anatomie der Hirnrinde. Upsala 1895.

³⁾ HELEN BRADFORD THOMPSON, The Total Number of Functional Nerve Cells in the Cerebral Cortex of Man, etc. Journal of Comparative Neurology. Vol. IX, p. 113—140. Granville, Ohio. 1899.

and the muscular system, we observe from the outset the greatest plasticity, adaptation, and morphological differentiation, in the world of Reptiles as in that of Mammals. But the central organ of animal life remains insignificant in Triceratops and Stegosaurus as in Diplodocus, it remains small in the Pterosauria, and it also remains at a low stage of organisation in the different eocene mammalian orders. Something of the same kind is seen in the existing animal world: equality in the brain quantity and the degree of animal organisation with great diversity in the other, the vegetative organisation.

According to the researches of GIUSEPPE LEVI and EDWIN CONKLIN the nerve-cells and the muscle-cells differ from all other cells, probably throughout the whole animal world, in this that they multiply only in the beginning of the individual's life. It is self-evident that the development, *i.e.* the phylogenetic growth, of the vegetative organs proceeds gradually, that of the animal organs, the brain and the whole of the nervous system with the muscular system, proceeds in leaps and starts.

This latter may then be imagined thus, that in species for the rest already differentiated in many directions, but not yet specialized in the central organ (and the whole of those systems), in the ontogenesis of some of the largest individuals, which possess *larger nerve cells*, according to the other law of the relations between brain quantity and body weight proportional as $P^{1/18}$, one nerve cell-division more takes place, — attended with further differentiation in the directions already started on.

Botany. — “On the Formation of “Triad”-groups of Chromosomes in the Divisions of the Nuclei of the Endosperm in *Mouriria anomala* Pulle.” By J. D. RUYS. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 3, 1924).

MONTGOMERY¹⁾ was the first zoologist who, in 1901, endeavoured to produce experimental evidence for the probability that at every fecundation the homologous paternal and maternal chromosomes attract each other and are linked so as to make up one pair. These pairs persist throughout the diploid phase, while at the meiosis each of the two chromosomes, migrating apart, moves towards the daughter-nuclei.

STRASBURGER²⁾ was the first botanist to demonstrate this for plants (1905) in *Galtonia* and *Funkia*. Numerous investigators have since found the same in many higher and lower plants. In several plants the chromosomes seem to exhibit a more or less pronounced affinity, whereas it is also true that many species have been examined in which the chromosomes display no such affinity.

The assumption of a certain degree of mutual affinity between male and female chromosomes would reasonably induce us to conclude that in the endosperm sets of 3 chromosomes must occur, because the endosperm originates from the fusion product of two polar nuclei and one male nucleus.

Already STRASBURGER²⁾ has been looking for these “triads”, which could be expected theoretically, but he has never been able to distinguish them. From this he concluded, that the affinity of the chromosomes is „saturated” every time after the union of two, and that a third chromosome cannot be linked to them.

¹⁾ T. H. MONTGOMERY, 1901. The spermatogenesis of *Peripatus* (*Peripatopsis*) *balfouri* up to the formation of the spermatids. Zool. Jahrb. Abt. Anat. u. Ontogen. Bd. 14, p. 277—368.

T. H. MONTGOMERY, 1901. A study of the chromosomes of the germcells of Metazoa. Transact. Amer. Phil. Soc. Vol. XX, p. 154—236.

²⁾ E. STRASBURGER, 1905. Typische und allotypische Kernteilung. Pringsheim's Jahrb. f. wiss. Bot. Bd. 42, p. 1—71.

³⁾ E. STRASBURGER, 1910. Chromosomenzahl. Flora Bd. 100, p. 398—446.

E. STRASBURGER, 1911. Kernteilungsbilder bei der Erbse. Flora Bd. 102, p. 1—23.

In "Das Problem der Befruchtungsvorgänge" B. NĚMEC, 1910, incidentally points out on page 118, that in divisions of the endosperm-nuclei of *Ranunculus Ficaria*, he once detected a set of 3 chromosomes. He does not believe that this points to an affinity such as STRASBURGER has tried to demonstrate but he thinks that degeneration or chromatin diminution comes into play here.

I am not aware of any other statements in the literature concerning triads in dividing nuclei of the endosperm.

It is true, in triploid plants several sets of 3 chromosomes were found in the vegetative nuclear divisions, e.g. by TAHARA ¹⁾ and OSAWA ²⁾ in *Morus*, by STOMPS ³⁾ in the bastard *Narcissus poeticus gigas* × *Tazetta*, by BELLING ⁴⁾, one of BLAKESLEE's co-workers, in a triploid *Canna*-bastard, but the occurrence of triads in the endo-



Fig. 1. *Mouriria anomala* Pulle.

Division of an Endosperm; the chromosomes are grouped into triads.

Oil-immersion $\frac{1}{12}$. Leitz Apert. 1.30.

Compensating Ocular 12. Length of tube 160 mm.

Drawings made at table-height with Camera lucida 2 ×.

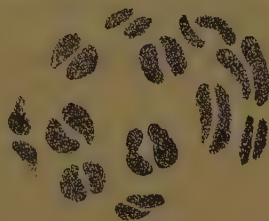


Fig. 2.

Mouriria anomala Pulle.

Vegetative nuclear plate: 24 chromosomes, grouped into dyads.

Oil-immersion $\frac{1}{12}$, Leitz Apert. 1.30.

Comp. Ocular 12. Length of tube 160 mm.

Camera lucida drawing at table-height.

¹⁾ M. TAHARA, 1910. Über die Kernteilung bei *Morus*. Bot. Mag. Tokyo. Vol. XXIV, p. 281—289.

²⁾ I. OSAWA, 1920. Cytological and experimental studies in *Morus*, with special reference to triploid Mutants. Bull. Imp. Sericult. Exp. Sta. Japan I, p. 317—369.

³⁾ TH. J. STOMPS, 1919. Gigas-Mutation mit und ohne Verdoppelung der Chromosomenzahl. Zeitschrift indukt. Abst. u. Vererb. Lehre. Bd. 21, p. 65—90.

⁴⁾ J. BELLING, 1921. The behaviour of homologous chromosomes in a triploid *Canna*. Proceedings of the National Academy of Sciences of the U.S.A. Vol. 7, p. 197—201.

sperm was restricted to the single case above-mentioned, i.e. of *Ranunculus Ficaria*.

When studying the development of the embryo-sac in some *Melastomataceae* I was so fortunate as to detect in *Mouriria anomala* Pullé a metaphase of a dividing endosperm-nucleus, in which the chromosomes were distinctly arranged in sets of 3. I counted 12 sets of 3, which agrees fairly well with the diploid number of chromosomes in a nucellus-nucleus, which number amounts to 24. (See figs. 1 and 2).

The chromosomes display in the diploid phase such an intense affinity, that it seemed to me at first as if only 12 could be counted. But on closer examination they proved to be dyads that looked exactly like gemini in a meiosis. This number, then, is exactly what we expected it to be.

In an anaphase of an endosperm nuclear division, in which the chromosomes passed to the poles in a random manner, we counted 72 chromosomes. Here the "triads" were no longer very conspicuous, but still quite distinguishable.

Utrecht, May 1924.

From the Botanical Laboratory.

Chemistry. — “*In-, mono- and plurivariant equilibria.*” XXVII.

By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of May 3, 1924).

Singular equilibria.

When in an equilibrium of n components in r phases:

$$E(n:r)(r') = A + B + \dots + L + M \dots + R \dots \dots \quad (1)$$

between the r' phases $L M \dots R$ a phases-reaction

$$\lambda_1 L + \lambda_2 M \dots + \lambda_{r'} R = 0 \dots \dots \dots (2)$$

may occur, then we shall call this equilibrium “singular” and we shall, as in (1), represent it by $E(n:r)(r')$. We call the r' phases, which participate in the phases-reaction, the “singular” phases; the $r-r'$ other ones, which do not participate in this reaction, the “indifferent” phases. Reaction (2) is called: the singular reaction.

When $r' = r$, then it passes into the equilibrium $E(n:r)$, formerly discussed, which is monovariant.

In equilibrium (1) only one singular part occurs, viz. $L M \dots R$, however, there are also equilibria with two and more singular parts. Before discussing more in detail those equilibria, we firstly shall indicate some examples.

I. *Equilibria with one singular part. $E(n:r)(r')$.*

In the equilibrium

a. $E(n:3)(2) = \text{solution} + (\text{ice} + \text{water-vapour}),$

just as we shall do in the following examples, the singular phases are placed between parentheses. It is viz. evident that in this equilibrium, independent on the number of components and on the composition of the solution, the reaction $\text{ice} \rightleftharpoons \text{water-vapour}$ may occur always.

The same is true for equilibria as:

b. $E(n:3)(2) = \text{solution} + (\text{solid benzol} + \text{vapour benzol})$

c. $E(n:3)(2) = \text{solution} + (\text{solid naphthalene} + \text{vapour naphthalene})$

etc. also for the equilibrium:

d. $E(n:4)(2) = A + \text{solution} + (\text{ice} + \text{water-vapour})$

wherein A may represent a solution, a mixed-crystal or a constant solid substance.

When we represent the two modifications of a solid substance F by F_α and F_β , then:

$$e. E(n. 3)(2) = \text{solution} + (F_\alpha + F_\beta)$$

$$f. E(n. 3)(2) = \text{vapour} + (F_\alpha + F_\beta)$$

$$g. E(n. 4)(2) = \text{solution} + \text{vapour} + (F_\alpha + F_\beta)$$

are equilibria, in which $F_\alpha + F_\beta$ is the singular part, independent on the fact how many components the liquid or the vapour contains.

When we represent by Z and $Z \cdot \alpha H_2O$ an anhydric substance and its hydrate, then the equilibria:

$$h. E(n. 4)(3) = \text{solution} + (\text{water-vapour} + Z + Z \cdot \alpha H_2O)$$

$$i. E(n. 5)(3) = A + \text{solution} + (\text{water-vapour} + Z + Z \cdot \alpha H_2O)$$

are also singular ones. Herein A may represent a liquid, a mixed-crystal or a constant solid substance.

In the examples, mentioned above, the singular part consists of constant phases only, viz. of phases, the composition of which is unvariable. There are, however, also equilibria, the singular part of which contains one or more variable phases.

This is f.i. the case in the equilibrium

$$j. E(n. 3)(2) = A + (\text{solid substance} + \text{liquid}),$$

when the liquid gets the same composition as the solid substance, independent on the fact whether this substance is mixed-crystal or it has a constant composition. A may represent a liquid, a vapour, a mixed-crystal or a constant solid substance.

The same is also true for equilibria:

$$k. E(n. 3)(2) = A + (\text{solid substance} + \text{vapour})$$

$$l. E(n. 3)(2) = A + (\text{mixed-crystal} + \text{liquid})$$

$$m. E(n. 3)(2) = A + (\text{mixed-crystal} + \text{vapour})$$

$$n. E(n. 3)(2) = A + (\text{liquid} + \text{vapour}),$$

when the two latter phases have the same composition, and also for the equilibria:

$$o. E(n. 4)(3) = \text{vapour} + (\text{liquid} + A_1 + A_2)$$

$$p. E(n. 4)(3) = \text{liquid} + (\text{vapour} + A_1 + A_2),$$

when the latter three phases have such composition that one of them can be formed from the two other ones.

II. *Equilibria with two or more singular parts.*

We shall represent equilibria with two singular parts by $E(n. r)(r')(r'')$ or by $E(n. r)S_2$. Those equilibria exist only then, when the phases of the two singular parts are different.

We might have examples of such equilibria f.i. in

g. $E(n.4)(2)(2) = (F_1 + F_2) + (\text{solution} + \text{vapour})$

r. $E(n.5)(2)(2) = (F_1 + F_2) + \text{solution} + (\text{ice} + \text{water-vapour})$

s. $E(n.6)(2)(3) = (F_1 + F_2) + \text{solution} + (\text{water-vapour} + Z + Z.\alpha H_2 O).$

Equilibria, of which two singular parts have one or more phases in common, have always three or more singular parts; in other words: when in an equilibrium two phases-reactions occur, which have one or more phases in common, then always more phases-reactions are possible. In previous communications¹⁾ we have seen that we can deduce all reaction-equations when two of them are known.

Consequently when in the equilibrium $A_1 + A_2 + \text{solution} + \text{vapour}$ a phases-reaction between A_1, A_2 and solution — and a phases-reaction between A_1, A_2 and vapour is possible, then it exists also between A_1 solution and vapour and between A_2 solution and vapour. Consequently we have an equilibrium:

t. $E(n.4)S_4 = ((A_1 + A_2 + \text{solution} + \text{vapour}))$

wherein 4 singular reactions are possible and which contains, therefore, 4 different singular parts. In similar cases we shall place the singular phases in parenthesis.

The equilibria $E(n.n+2)$ viz. equilibria of n components in $n+2$ phases belong also to this kind. As in general in those equilibria $n+2$ singular reactions may occur, they are equilibria $E(n.n+2)S_q$ in which $q = n+2$. In special cases, as we have seen previously²⁾ q can be also smaller than $n+2$; further we still shall refer to an example.

In this specimen of equilibria there may occur also phases, which take part in no single phases-reaction and which are consequently indifferent to all reactions. When we represent by F_α, F_β and F_γ three modifications of a solid substance F , then is

u. $E(n.4)S_3 = \text{liquid} + ((F_\alpha + F_\beta + F_\gamma))$

a similar equilibrium. Herein are possible the three singular reactions $F_\alpha \rightleftharpoons F_\beta$, $F_\alpha \rightleftharpoons F_\gamma$ and $F_\beta \rightleftharpoons F_\gamma$, but the liquid is indifferent for each of those three reactions. Later we shall discuss a such-like example.

¹⁾ F. A. H. SCHREINEMAKERS; these Proceedings XXIV 882 (1915); 1100 (1916).

²⁾ In previous communications [Proceedings XXV 535, 627, (1916), 754, 860, 939, 1102 (1917)] already several invariant equilibria with indifferent and singular phases have been discussed. In some cases we have even supposed a very special singularity.

Equilibria with one singular part.

We now shall consider more in detail the equilibria, in which only one singular part occurs. When we take the singular part $LM \dots R$ of the equilibrium (1) for itself alone, then we have an equilibrium in r' phases, between which a phases-reaction is possible. Formerly we have seen that such an equilibrium is monovariant and is represented in the PT -diagram by a curve. We call this curve "the PT -curve of the singular part" or also "the singular PT -curve".

We now shall deduce the following property:

an equilibrium $E(n.r)(r')$ is represented in the PT -diagram either by its singular PT -curve or by a point of this curve.

In order to deduce this we imagine that an arbitrary equilibrium E passes through all states, which permit the kind of this equilibrium; to each of those states belongs a definite T and P . Those form in a PT -diagram the PT -area of the equilibrium E . This area may be a field with one or more leaves, a curve, or also a point.

We now take away from E in each state in which it may be found at every time one or more definite phases, but of course repeatedly the same ones; at every time there remains an equilibrium or phases-complex E' . Consequently with each state of E a state of E' is corresponding; everywhere where the region of E is found, we, therefore find also the region of E' . But, as there may be perhaps also states of E' which do not belong to a state of E , the area of E' may perhaps extend itself further. Consequently we may say:

when we take away from an equilibrium one or more phases, then the new PT -area is extended over the first one or it coincides with it.

Now we apply this to the equilibrium $E(n.r)(r')$; when we take away from this the $r-r'$ indifferent phases, then an equilibrium $E(n.r')(r')$ arises, viz. the singular part. This is monovariant, as between all phases a phases-reaction is possible; it is represented in the PT -diagram by a PT -curve.

It now follows from the previous that this PT -curve must extend itself over the area of the equilibrium $E(n.r)(r')$ or that it must coincide with this. The equilibrium $E(n.r)(r')$ is, therefore, either a point of this PT -curve or it coincides with it.

When the equilibrium $E(n.r)(r')$ is represented by its singular PT -curve, then consequently P and T are dependent on one another.

We then may choose or T or P as independent variable ones, but not both at the same time. Yet, as we shall see further, it may have still different other freedoms.

When $E(n, r)(r')$ is represented by a point of its singular PT -curve, then consequently P and T are completely determined.

We may precise more in detail these results in the following way. We take at T_0 and under P_0 an equilibrium:

$$E_0(n, r)(r') = A_0 + B_0 \dots + L_0 + M_0 \dots + Q_0 + R_0 \dots \quad (3)$$

in which $A_0, B_0 \dots$ represent the $r-r'$ indifferent phases and $L_0 \dots R_0$ the r' singular phases. We represent the phases-reaction in the singular part by:

$$\lambda_1 L_0 + \lambda_2 M_0 \dots + \lambda_{r'} R_0 = 0 \dots \dots \dots (4)$$

We now take at $T_0 + \Delta T$ and $P_0 + \Delta P$ an equilibrium:

$$E(n, r)(r') = A + B \dots + L + M \dots + Q + R \dots \dots \dots (5)$$

the phases of which differ infinitely little from those of the equilibrium E_0 . In a similar way as in (4) we may represent the phases-reaction.

In order to represent the compositions of the phases of the equilibrium E we take n composants. For this we choose the $r-r'$ phases $A, B, \dots$; of the r' phases $L, M, \dots, R$ we may choose $r'-1$ phases only, as those are dependent on one another in accordance with (4). Consequently we take still $n-r+1$ free composants $X, Y, \dots$. We represent the composition of an arbitrary indifferent phase I by:

$$I = x X \dots + a A_0 \dots + m M_0 \dots + q Q_0 + \varrho R_0 \dots \dots \dots (6)$$

In order to represent the $r-r'$ indifferent phases $A, B, \dots$ of E , we give in (6) to the variables successively the indices $1, 2 \dots (r-r')$.

We shall represent an arbitrary singular phase S by:

$$S = x' X \dots + a' A_0 \dots + m' M_0 \dots + q' Q_0 + \varrho' R_0 \dots \dots \dots (7)$$

In order to represent the r' singular phases $L, M, \dots$ of the equilibrium E , we give in (7) to the variables the indices $1, 2 \dots r'$.

It is not necessary now that each of the parts of the singular equilibrium contains the n composants. We see f.i. in the examples $\alpha-g$ that the singular part contains one composant only, in examples h and i that it contains two composants. When we imagine in example j that A represents also a solid substance, then the singular part contains all composants and the indifferent part contains only one.

We now shall assume that these are n_g composants which occur in both parts of the equilibrium; further that there are n_i composants

which occur in the indifferent part only. We call them "indifferent composants" (not: the composants of the indifferent part, for this number is $n_g + n_i$). Further there may be n_s composants, which occur in the singular part only. Consequently we have $n_g + n_i + n_s = n$.

A similar equilibrium may sometimes exist only under very definite conditions, which we shall discuss later. We shall assume here that it exists. Here must be noticed that it may exist always when there is at least one variable phase, which contains all composants and we may assume that this is the case with a liquid. In that case n_i or n_s are zero then. We now shall divide the equations for the equilibrium E into three groups.

The first group relates to the indifferent part, taken for itself only. This is an equilibrium of $n_i + n_g$ composants in $r - r'$ phases; it has, therefore, $n_i + n_g - (r - r') + 2$ freedoms. When we call i the number of variables, which occur in the indifferent part, then, consequently, we may say also that between TP and those i variables there exist $i - (n_i + n_g) + (r - r')$ equations.

The second group relates to the singular part, taken for itself only. This equilibrium is monovariant. When we call s the number of variables which occur in this singular part, then, consequently, there are $s + 1$ equations between PT and those s variables. When we take, therefore, T as independent variable, then P and the s other variables are, consequently completely defined. Consequently this equilibrium is represented by a curve in a PT -diagram.

The third group of equations expresses that the indifferent and the singular part are in equilibrium with one another.

We now assume that as well in the indifferent part as in the singular part there is a variable phase; the first one then contains $n_i + n_g$ composants, the other one $n_s + n_g$ composants. It is sufficient now for the equilibrium that both those phases are in equilibrium with one another. We then still get n_g equations.

Then we have $[i - (n_i + n_g) + r - r'] + (s + 1) n_g$ equations between the $i + s + 2$ variables; consequently the equilibrium has $n_i - (r - r') + 1$ freedoms.

In the following we shall represent n_i viz. the number of indifferent composants by N . When we put $r - r' = R$ then R represents, therefore, the number of indifferent phases. We now shall call an equilibrium $E(n, r)(r')$, in which occur N indifferent composants and $R = r - r'$ indifferent phases, an equilibrium $E(N, R)$. Then we find:

the singular equilibrium $E(N, R)$ has $N - R + 1$ freedoms.

This result is not in accordance with the phase-rule of GIBBS. The number of freedoms is viz. generally smaller than $n-r+2$ and we are limited in the choice of those freedoms in so far, that we may choose T or P as independent variable, but not both together.

This deviation of the phase-rule is however only apparent, in so far that this limitation finds its cause in the conditions which we have attributed to the equilibrium. In a following communication we shall refer to this.

This result is in accordance also with that deduced in a previous communication viz. that an equilibrium $E(n, r)(r)$ has one freedom only. In a similar equilibrium is viz. $N=0$ and $R=0$ so that $N-R+1$ becomes $=1$.

We may assume also that all phases of the singular part have a constant composition, this is a. o. the case in the examples $a-h$.

The second group of equations is reduced then to the one equation:

$$\lambda_1 \zeta_L + \lambda_2 \zeta^M \dots + \lambda_{r'} \zeta_Q = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In order to find the third group we must express that one of the variable indifferent phases is in equilibrium with each of the r' singular phases. As, however, in accordance with (4) those are dependent on one another, it is sufficient to take $r'-1$ of these phases; consequently we get $r'-1$ equations.

As in this case $n_s=0$, follows $n_i + n_g = n$; further is $s=0$. The total number of equations becomes therefore $i-n+r$, while there are $i+2$ variables consequently the equilibrium has $n-r+2$ freedoms.

This result is in accordance with the number of freedoms following the phase-rule of GIBBS; the only limitation still is that we may not choose P and T at the same time as independent variables. It is also in accordance with the $N-R+1$ freedoms, which must have an equilibrium $E(N, R)$; the number N of the indifferent composants is viz. $n-(r'-1)$; the number R of the indifferent phases is $r-r'$; consequently $N-R+1 = n-r+2$.

It could still also be assumed that the $r-r'$ indifferent phases have all a constant composition. Then the equilibrium can exist, however, only when some conditions are satisfied; we shall refer to this later. When it is not a phases-complex but an equilibrium, then it is invariant.

We may resume the previous in this way.

An equilibrium $E(N, R)$, consequently, an equilibrium with one singular part, with N indifferent composants and R indifferent phases, has $N-R+1$ freedoms.

When $N-R+1 > 0$ then it is represented in a PT -diagram by its singular PT -curve. In each point of this curve the singular part is completely defined, but the indifferent part has still $N-R$ freedoms.

When $N-R+1 = 0$ then it is represented by a point of this curve.

Consequently it is apparent from the previous that some of the equilibria with one singular part are represented by their singular PT -curve, other ones by a point of this curve. Now it is the question which point of the curve it is.

This case may not occur at an equilibrium of which the singular part is constant; as this has $n-r+2$ freedoms, r should be $=n+2$. Then, however, more than one phases-reaction is possible. Consequently we may expect this case only with equilibria, of which the singular part contains also variable phases.

In order to examine this we take the equilibrium $E_*(n.r)(r')$ which we have represented in (3) and in which phases-reaction (4) occurs. We now assume that at other temperatures and pressures there exists an equilibrium, as we have represented in (5); we suppose, however, that this is not an equilibrium $E(n.r)(r')$ but an equilibrium $E(n.r)(r)$. In other words we assume that between the r phases a phases-reaction may occur, which we shall represent by

$$\mu_1 A + \mu_2 B \dots + \lambda_1 L + \lambda_2 M \dots + \lambda_r R = 0 \quad . \quad . \quad (9)$$

This equilibrium is monovariant; in a P, T -diagram it is represented by a curve, the direction of which is defined in every point by

$$\frac{dP}{dT} = \frac{\sum (\mu H + \sum (\lambda \eta))}{\sum (\mu V) + \sum (\lambda v)} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When this equilibrium proceeds along its PT -curve, then the reaction-coefficients in (9) change continuously. We now assume that in a definite point i all coefficients become zero; we then get the singular equilibrium (3) in which reaction (4) may occur. When we take on this curve points in the vicinity of i , then μ_1, μ_2 , etc. get infinitely small values.

The equilibrium (5) [viz. $E(n.r)(r)$ as we have assumed] is converted therefore, in point i into the equilibrium (3). This can take place, however, only then, when the singular part contains also variable phases. When viz. all phases were constant, then reaction (4) should be valid also always in (9) and there should be possible more than one phases-reaction.

equations the $q-2$ other ones. The directions of the q singular PT -curves in their point of intersection i are, therefore, also dependent on one another.

It shall not be necessary to show fully that equilibria with more than 2 singular parts, which are independent on one another, do not exist.

Then we get for the definition of T and P more than two equations which are independent on one another, or in other words: we have in the PT -diagram more than two arbitrary PT -curves. It is evident that only very occasionally they shall go through one point.

(To be continued).

Leiden. Lab. of Inorg. Chemistry.

Physics. — “*Perspective shortening in spectroheliograms, and transparency of the photosphere*”. By Prof. W. H. JULIUS.

(Communicated at the meeting of May 31, 1924).

When studying some H_{α} -spectroheliograms obtained on the very few days of clear sky and good seeing we were favoured with in Utrecht during the spring of 1924, I was struck by a remarkable feature of the solar image, which, though never to my knowledge mentioned before, then proved also to be observable on a great many published spectroheliograms of Mount Wilson and Meudon, and seems to indicate an important general property of the photosphere.

The phenomenon in question may be briefly described as follows: “the perspective foreshortening towards the limb, which the irregular pattern or network of flocculi would show if it were due to some bright-and-dark structure confined to a spherical shell of less than a thousand kilometers thickness — that foreshortening does *not* appear.”

Suppose the photosphere really were comparable to a layer of clouds (as some theories still sustain), or were composed of gaseous matter so quickly increasing in density with depth (after SCHWARZSCHILD, EMDEN, ABBOT, and others) as to make a layer of a thousand kilometers thickness entirely opaque to direct radiation, what would be the character of the image if the spherical surface were divided into irregular bright and dark patches showing average dimensions of, say, 15000 kilometers ($\frac{1}{100}$ solar diameter)?

The answer is best given by referring to the plate, fig. 1, *a*, which represents the perspective aspect of nine squares, each 10° side, arranged on a quarter of a sphere whose centre lies behind *C*. The elements of an irregular pattern, if present within those squares, would appear foreshortened in the proportion of the apparent sides of the squares they belong to. An example of such an aspect is shown by fig. 1, *b*, where a belt of the moon is represented on the same scale as fig. 1, *a*. The craters, circular near the centre of the disc, are elsewhere ellipses with axes in the right proportion and position, according to their distances from the limb.

Now compare with this the H_{α} -spectroheliograms fig. 1, *c* (Meudon, Aug. 19 1909, $10^{\text{h}} 42^{\text{m}}$) and fig. 1, *d* (Utrecht, March 14 1924,

14^h 8^m), both typical of a quiet, spotless condition of the sun. It is very striking, when once noticed, that the perspective foreshortening towards the limb, proper to a spherical surface, does not appear on these photographs. A similar impression is made by all H_{α} -spectroheliograms showing sufficient detail near the limb. There is some shortening visible, but to an extent much less than would correspond to the perspective appearance of a stained spherical surface. Indeed, even close by the edge several structure elements show much the same berry-like shape as near the centre. If we isolate an excentrically situated circular or rectangular piece of such a spectroheliogram on which no part of the limb appears, it is almost impossible to find out the probable position of the limb by merely inspecting the structure¹⁾. In a section of the moon's image the corresponding problem would present no difficulty.

Let us examine how far existing interpretations of spectroheliograms are able to account for the peculiarity in question.

According to the original explanation given by HALE²⁾ and DESLANDRES bright calcium flocculi are strongly radiating masses of calcium vapour suspended in a medium of smaller emissive power for the same radiation. If such self-luminous clouds are thin, extensive, practically flat objects, they will show the regular foreshortening near the limb; if their shape is more globular, they will seem to overlap at the sun's edge, the darker interstices will disappear as we approach the limb. Neither inference, however, corresponds to the observed aspect of the marginal regions.

It is usual to describe the hydrogen spectroheliograms somewhat differently: there, the name "flocculi" is commonly applied to the *dark* places, as these very often (not always) coincide with bright calcium flocculi³⁾. Interpreting those dark hydrogen flocculi as absorbent clouds we meet with a similar difficulty: we should expect them to overlap near the border, and to prevent the brighter interstices from being visible, but they do not.

Another, entirely different interpretation of spectroheliograms — which I proposed in 1904⁴⁾ — attributes the peculiar distribution

¹⁾ It is interesting to examine in this connection the plates 52 and 53 of the *Annales de l'Observatoire d'Astronomie physique de Paris*, Tome IV, where parts of H_{α} -spectroheliograms of March 28 and Aug. 8, 1910 are reproduced, the positions of which on the disc are indicated on plate 55.

²⁾ G. E. HALE and F. ELLERMAN. *Publications of the Yerkes Observatory*, Vol. III, Part I, 1903.

G. E. HALE, *The Study of Stellar Evolution*, p. 89 (Chicago, 1908).

³⁾ *Ibid.* p. 95.

⁴⁾ These *Proceedings* Vol. VII, p. 140, 1904; *Astroph. Journ.* 21, 278, 1905.

of the light in such images only for a small part to emission by the gas, a line of which is chosen, but principally to anomalous refraction of waves belonging to the continuous spectrum, on their passage through the irregular gradients of optical density in a rare, transparent medium. This view of the subject suggested, as an illustration, the following simple experiment.

A 75-watt Philips argenta-lamp (of milky-white glass), 8 cm.



Fig. 2.

diameter, is concentrically placed inside a thin glassbulb of 11 cm. diameter, and the intervening space (cf. fig. 2) is filled up with irregular glass grains of 1 to 3 mm. average dimension, embedded in water. The whole is then submerged in a rectangular glass tank filled with water, and the lamp lighted. We thus have a uniform source of light surrounded by a rather thick, perfectly translucent but not transparent spherical shell in which the light is scattered by refraction. Fig. 1,*e* is a photograph of a belt of the illuminated bulb, intentionally slightly out of focus because the

experiment only aims at producing an irregular distribution of the light¹⁾. This image too shows but little foreshortening, as will be readily understood considering the transparency of every single grain. That, nevertheless, there is some effect visible like foreshortening at the edge, is chiefly due to the onesided position of the source of light. Indeed, by plunging a second lighted argenta-lamp into the tank, so as to illuminate the "limb" from the other side as well, we are able to reduce the effect.

If we ignore the sharpness of the boundaries and edges of the grains in our experiment, we really see some resemblance between fig. 1,*e* and the spectroheliograms. The otherwise enigmatic deficiency of foreshortening in the latter may, therefore, be explained on the basis of the anomalous dispersion theory of Fraunhofer lines and of spectroheliograms. We only have to assume the photospheric matter

¹⁾ To produce a fairer analogy with the sun it would have been preferable to avoid reflexions and sharp edges by taking, instead of glass grains, some substance like gelatine or sago grains, the refractive index of which would pass gradually into that of water; but it is difficult to obtain similar material in a sufficiently translucent state, and to keep it unaltered for a reasonable time.

to be very transparent for all waves ¹⁾ — except for those belonging to the extremely narrow cores of the Fraunhofer lines — but honeycombed with irregular gradients of *optical* density. (This property or condition is not incompatible with low mechanical density, as it depends on the relative numbers of atoms in high quantum states, which may be great under circumstances prevailing in the sun. — For the present, however, we only draw attention to the *optical* analogy between our experiment and the solar phenomenon, and leave a closer consideration of the required and possible physical conditions in the sun for later investigation).

In order to obtain a numerical expression for the extent to which some perspective shortening yet appears, we have drawn, on paper prints of the spectroheliograms, radii of the disc in four arbitrary directions and divided each radius into six equal parts. The numbers of flocculi cut by the consecutive parts were then counted. The averages of the numbers thus found in the successive concentric zones of the disc are given in Table I — transposed, however, in such a way that the number for the central zone was always taken as unity. Each number in the table (except in the last column but one) therefore represents the mean of four numbers found in different directions ²⁾. The first four columns refer to H_{α} -spectroheliograms obtained in Utrecht; the next four to H_{α} -plates of Meudon ³⁾. Then follow the results of similar countings made on a calcium K_{α} -spec-

TABLE I.

	Utrecht H_{α} 136	Utrecht H_{α} 133	Utrecht H_{α} 126	Utrecht H_{α} 134	Meudon H_{α} 12-8-09	Meudon H_{α} 12-8-09	Meudon H_{α} 12-8-09	Meudon H_{α} 11-9-09	Meudon Ca, K_{α} 7-5-10	Perspective shortening	Model with glass grains.
$0-1/6$ R	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$1/6$ R— $2/6$ R	1.00	0.96	1.05	1.12	1.06	1.03	1.09	1.15	1.07	1.02	1.16
$2/6$ R— $3/6$ R	1.05	1.07	1.13	1.09	1.03	1.06	1.16	1.05	1.18	1.10	1.11
$3/6$ R— $4/6$ R	1.17	1.14	1.05	1.00	1.00	1.09	1.18	1.18	1.35	1.24	1.16
$4/6$ R— $5/6$ R	1.21	1.11	1.31	1.24	1.13	1.18	1.18	1.38	1.27	1.52	1.25
$5/6$ R— $6/6$ R	1.34	1.25	1.67	1.49	1.26	1.24	1.36	1.80	1.79	3.50	1.41

¹⁾ Molecular scattering will be considered further on.

²⁾ The counting has been performed partly by Dr. MINNAERT, partly by myself.

³⁾ These plates are from a collection of copies on glass which Mr. DESLANDRES had the kindness of sending me several years ago.

troheliogram, and, in the last column, of countings relating to our glass-grain model.

The last column but one, finally, gives the proportions that would have been found in the successive parts of a radius if the flocculi were flat objects extending over a spherical surface, or nearly flat ones confined to a thin opaque layer.

Table II contains the average results for the H_{α} -plates of Utrecht and of Meudon separately, and also the general mean. The latter

TABLE II.

	Utrecht (H_{α})	Meudon (H_{α})	General mean
$0-\frac{1}{6} R$	1.00	1.00	1.00
$\frac{1}{6}-\frac{2}{6}$	1.03	1.08	1.05
$\frac{2}{6}-\frac{3}{6}$	1.08	1.08	1.08
$\frac{3}{6}-\frac{4}{6}$	1.09	1.11	1.10
$\frac{4}{6}-\frac{5}{6}$	1.22	1.22	1.22
$\frac{5}{6}-\frac{6}{6}$	1.44	1.42	1.43

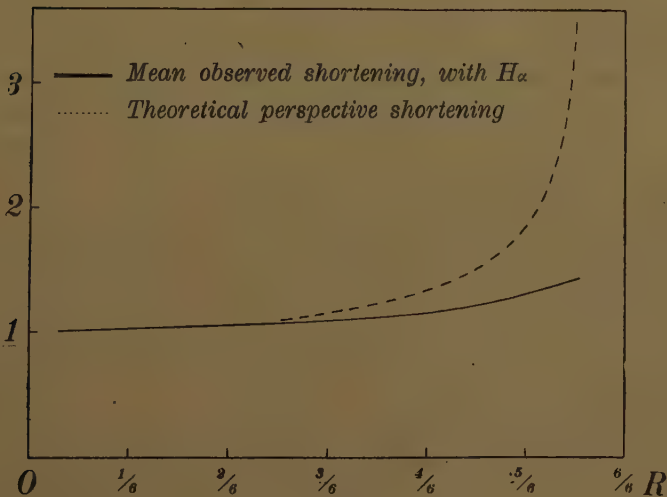


Fig. 3.

is represented by the full curve in fig. 3, where, moreover, the broken curve corresponds to the numbers of the last column but one of Table I.

Near the limb the observed shortening evidently lags far behind the theoretical value proper to a spherical surface.

Similar countings have also been made on curves (obtained with a microphotometer) showing the intensity distribution along a diameter of the spectroheliogram. The general result was the same.

Basing ourselves on the average size of the flocculi we may roughly estimate a minimum depth of the translucent shell in which molecular scattering and black-body radiation are still neglectable.

Indeed, considering the practical absence of foreshortening towards the limb we see no reason for assigning to the bright elements of the structure smaller average dimensions in the direction of the sun's radius than perpendicular to it. Now examine fig. 4¹⁾. This H_{α} -spectroheliogram has been obtained by ELLERMAN on Mount Wilson, with excellent apparatus and evidently under very favourable conditions. As the contrasts scarcely decrease on approaching the limb, the flocculi (bright or dark) cannot be embedded in a "mist"; they are visible to their full extent in the direction of the line of sight. This means that molecular scattering does not play a prominent part in the phenomenon. On the large sunspot a white circle has been traced showing the size of the earth, nearly 13000 km. diameter. Comparing this with the average size of the flocculi we conclude, that we look into the sun at least as far as 13000 km., and probably much deeper.²⁾

What, then, is a possible interpretation of the bright-and-dark structure we see?

When making our Utrecht spectroheliograms the second slit had been set on the red or the violet edge of the H_{α} -line (whose average width is 1 ÅNGSTRÖM unit). The slit was usually 0,6 Å wide and its middle was 0,5 Å distant from the centre of the line; so the true absorption line fell just outside the slit. DESLANDRES' photographs have been made under similar conditions. According to the dispersion theory, therefore, only a very small proportion of the light used can have been due to selective emission by the rare hydrogen; the bulk must belong to the continuous spectrum. So we have to inquire into the origin of that continuous light.

The great transparency of the matter composing the outer, rather thick, layer just beneath the photospheric level excludes the idea that black or grey radiation or grey absorption by that very matter should play any appreciable part in the phenomena of the photo-

¹⁾ Reproduced from a book by G. E. HALE: "The New Heavens", with the kind permission of the author, to whom we feel very much obliged.

²⁾ A stereoscopic investigation of spectroheliograms — now in progress — leads to results in good harmony with this conclusion.

sphere. The only remaining explanation, therefore, seems to be that the true source of the radiation is deep-seated, and that it is a nearly pure effect of irregular raycurving if we get the impression that white light is radiated even by the outermost zones of the disc.

In the marginal parts of the disc bright flocculi would then in general indicate places where there is much scattering by refraction, whereas in the central parts such places might, on the contrary, show darker than their surroundings. (This statement applies to regions with "micro-structure", the elements of which are too small to be separately visible from the earth — probably the common case. In cases of "macro-structure" the position of the principal gradients with respect to the incident light and to the observer has of course to be taken into consideration, as, for instance, in the disturbed regions surrounding sun-spots).

It is very remarkable that in spectroheliograms, made with light of one of the calcium lines H or K , perspective shortening is more apparent than in H_{α} -images (Cf. fig. 1, f'). This may be due to the fact (which depends on molecular scattering) that with violet light we are looking less deep into the sun than with red light. The transparent refracting shell is thinner, its mean level lies higher for violet than for red; we therefore may expect the elements of the refracting structure to be more extensive on the average (because the steepness of the irregular gradients will in general decrease outwards); and as those larger flocculi are moreover confined to a thinner layer, the condition in some degree approximates to that of a spherical surface.

Another peculiarity in which solar images in H - or K -light differ from those in H_{α} -light is connected with the local presence of the bright reversals H_2 and K_2 in the spectrum¹⁾. In the spectroheliograms the reversals manifest themselves as a kind of network spread all over the disc. Whatever may be the nature of those bright regions — if they are characteristic of a special condition prevailing at a certain *level* in the sun, it is evident that they must show the perspective shortening typical of a rather thin spherical layer. The difference in behaviour between these calcium- and the hydrogen images may thus perhaps find an explanation.

Summarizing we conclude that the deficiency of perspective shortening observed in the marginal parts of spectroheliograms evidences high transparency of the photosphere to the light of the

¹⁾ Similar reversals occur in H_{α} , but rarely, and only in the neighbourhood of sun-spots or in highly disturbed regions.

continuous spectrum. Grey radiation can, therefore, not be emitted by the outer layers of the photosphere to any appreciable amount. Nevertheless white light emerges from every point of the disc up to the edge. A reasonable way to explain this seems to be, that we are dealing with radiation coming from profound layers, and scattered towards the earth by irregular gradients of optical density in a rare medium.

There are no sufficient grounds for the assertion often made, that the solar gases cannot sensibly refract the light; the recent development of the atomic theory furnishes us with starting points for inquiring into the circumstances, under which a strongly irradiated mixture of rare gases may become sufficiently heterogeneous for giving appreciable deviations to beams of light.

I am very much indebted to Dr. M. MINNAERT for his valuable assistance throughout this investigation.

Heliophysical Institute.

Utrecht, May 1924.

Palæontology. — "*Figures of the Calvarium and Endocranial Cast, a Fragment of the Mandible and three Teeth of Pithecanthropus Erectus.*" (Plates I to XI). By Prof. EUG. DUBOIS.

The 32 photographic figures hereby published, bearing on all the fossil remains of *Pithecanthropus erectus* hitherto discovered except the femur, are destined for a future memoir. However, an illustration of the report "On the Principal Characters of the Cranium and the Brain, the Mandible and the Teeth of *Pithecanthropus Erectus*" (in this Volume of the Proceedings, pp. 265 to 278) appearing to be a desideratum, and the importance of the subject requiring even in a brief report the same accurate mode of representation as in an extensive treatise, an anticipating publication of the figures destined for the memoir was necessary. I intend to present soon another report, on the femur, illustrated in the same way. The figures of the original fossils shall, moreover, be found of great aid, I expect, in the study of the casts.

As to the mode of representation of the fossil remains the following remarks may be placed here.

All the photos are taken with the aid of the "stereorthoscope", of my own contrivance (1901), to be described as a cubiform wooden or metallic frame bearing a network, in five sides of the cube, of equidistant orthogonally crossing strings, the sixth side being open, and the whole forming a "cage", in or behind which the object, such as a skull, can be placed according to and photographed exactly in certain planes, such as the natural median plane, the transversal plane of the glabella-inion line, etc.

The photos natural size of the calvarium (figures 1 to 6) are taken with a lens of 3 m. focal distance, consequently with the object placed at 6 m. from the lens. In this way the figures are nearly geometrical projections.

The photos natural size of the endocranial cast (figures 7 to 9) are taken with a lens of 1.50 m. focal distance, consequently at 3 m. distance between the object and the lens. Even these figures do only little depart from geometrical projections.

The same is true concerning the smaller objects: fragment of the mandible and teeth, although taken with a lens of smaller focal distance (0.60 m.). The figures of these objects are enlarged to $2 \times$ natural size.

EXPLANATION OF THE PLATES.

PLATE I.

Fig. 1. — Calvarium, from above. Transversal glabella-inion plane.

Extensive corrosion of the surface. The white-spotted areas behind the coronal suture, on both sides of the sagittal suture, are places where the outer wall of the parietal bones has entirely disappeared and the diploë laid bare. The same at the much worn off left external angular process of the frontal bone. The right process broken off in an oblique line. The natural constriction of the frontal is only conserved on the right side. Evident trigonocephalism. The obliquity of the occiput is due to corrosion. The smooth area on the hinder part of the parietal bones is artificial.

PLATE II.

Fig. 2. — Calvarium, from the left side. Sagittal (median) glabella-inion plane.

This side, especially in the temporal fossa, deeply corroded, in some places nearly to the vitreous table, at the border even to perforation.

Fig. 3. — Calvarium, from the right side.

The loss of bony substance is not so large as on the left. The fracture surface of the external angular process is very clear. The deepest part of the temporal fossa 45 to 50 mm. from the glabella. Sharp bending of the parietal surface, at about the place of the temporal line, chiefly due to the loss of the outer table of the bone on the side of the sagittal suture. On the border of the calvarial fragment, 6 cm. before the torus occipitalis, a very compact but externally much corroded part of the petrosum, 1 cm² in extent, immediately before which the place of the external auditory meatus must have been. At and near the occipito-mastoid suture furrows and parts of canals for the blood vessels; the foramen mastoideum on the border line. On comparing the external outline of the borderwards increasingly defective nuchal part of the occipital bone, in figures 2 and 3, with the outlines of the intact inner surface of that part, in figures 8 and 9, it becomes evident that the angle of the nuchal part is not human but pithecoïd.

PLATE III.

Fig. 4. Calvarium, from before. Front view. Perpendicular to the transversal glabella-inion plane.

Glabellar region defective and corroded. On the right side, 28 mm. from the middle, a supraorbital foramen. The swelling at the bregma exaggerated by the mentioned loss of the outer table of the parietal bones. Keel of the frontal bone.

Fig. 5. — Calvarium, from behind. Occipital view. Perpendicular to the same plane as in fig. 4.

Inion evident, but still a little bony substance lost here.

PLATE IV.

Fig. 6. — Calvarium, from below. Transversal glabella-inion plane.

Wide and deep frontal sinus; the right half wider than the left one, the incomplete septum encroaching on the line of the median crista galli. The



Fig. 1.

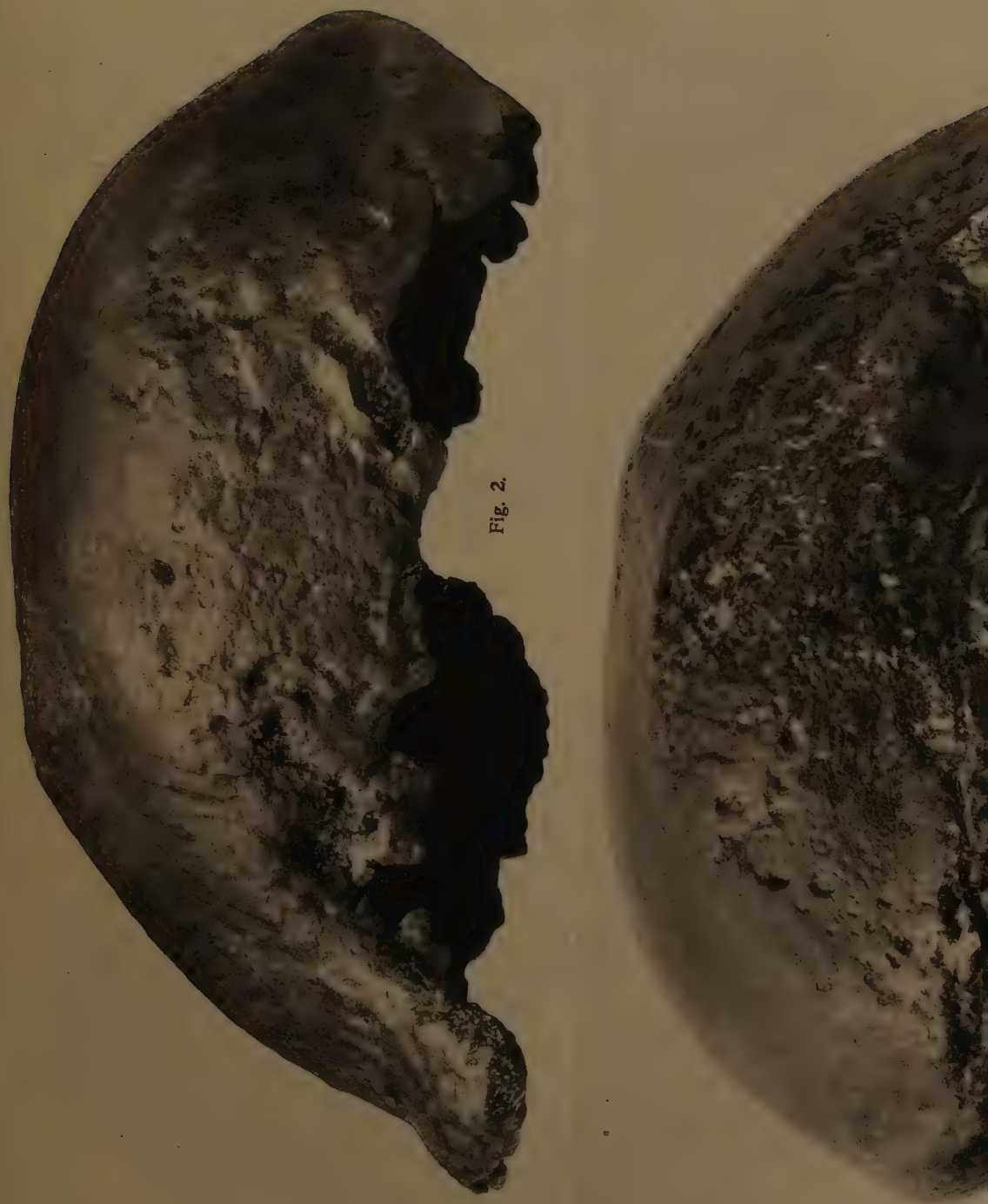


Fig. 2.



Fig. 4.



Fig. 5.

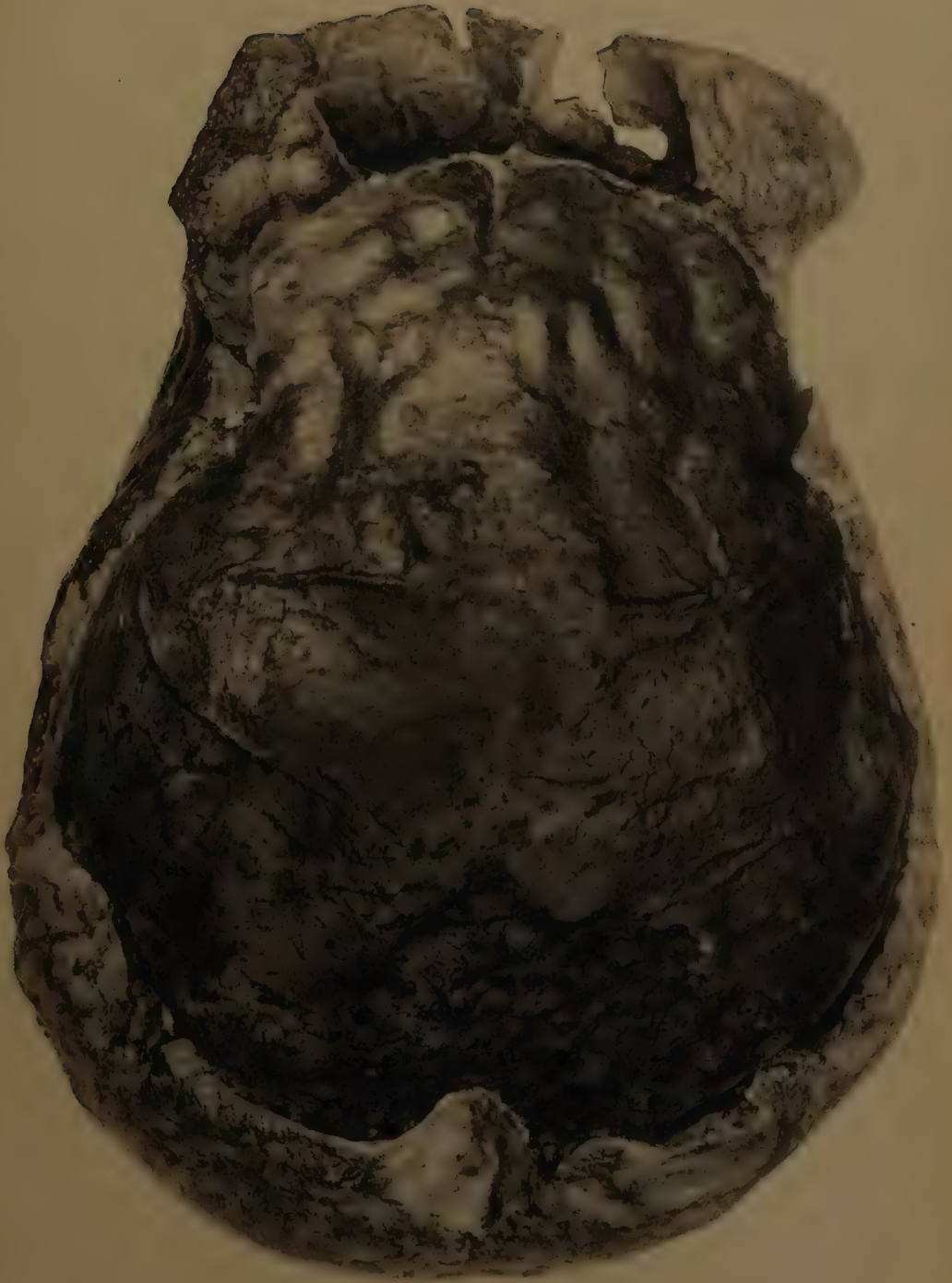


Fig. 6.

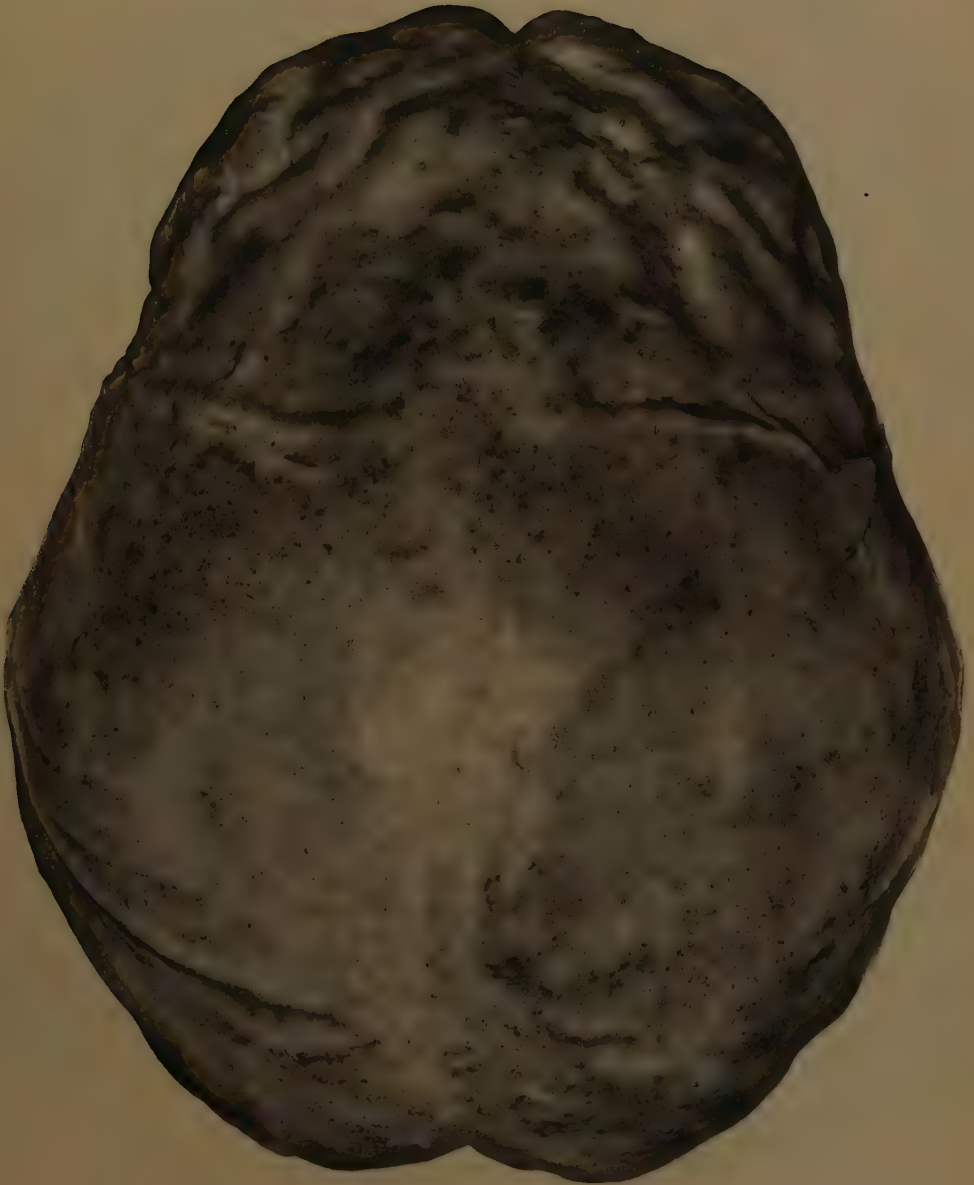


Fig. 7.



Fig. 8.



Fig. 9.



Fig. 10.

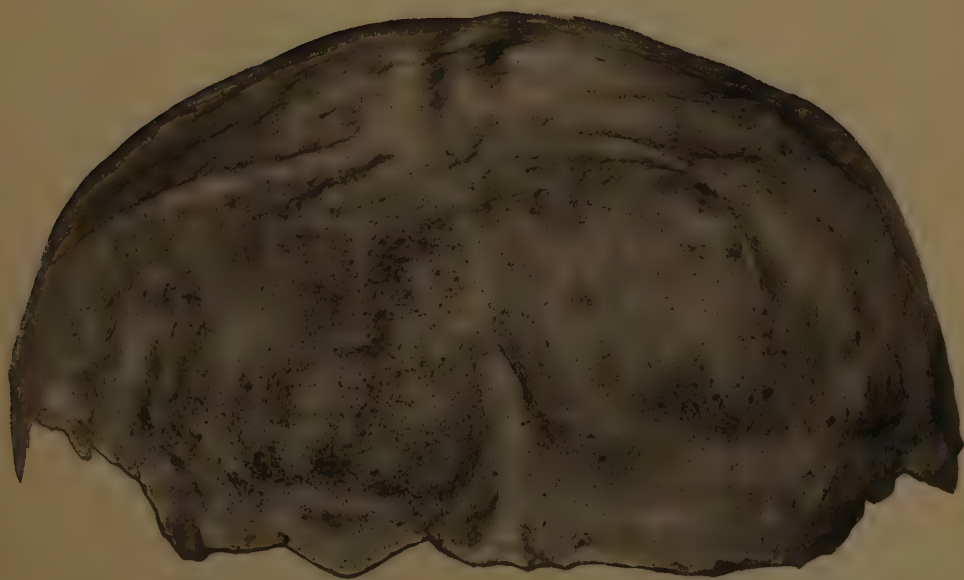


Fig. 11.

photo having been taken before the endocranial surface was entirely laid bare, some stony matter still existent in the occipital region. Strong impressions of the frontal convolutions and the ramification of the arteria meningeae media on both sides. Sharp edged middle part of the occipital border by the baseward increasing loss of bone.

PLATE V.

Fig. 7. — Endocranial cast from above. Transversal plane of the cerebral fronto-occipital axes. Effect of trigonocephalism on the brain form.

Exceedingly strong frontal convolutions. Evident intraparietal sulci. Sulcus lunatus manifest, particularly on the right side. Sinus sagittalis and ramification of the arteria meningeae media on both sides. Sulcus frontalis medius particularly developed as fronto-marginalis causing a notch, on both sides, in the frontal outline, about 20 mm. from the middle line.

PLATE VI.

Fig. 8. — Endocranial cast. Left side.

Ramification of the stronger ramus anterior and the less strong ramus posterior of the arteria meningeae media. Sulcus frontalis superior 3 to 4 mm. from the frontal outline. Sulcus frontalis medius, clearly homologue with the sulcus rectus or principalis of the Apes, 8 to 10 mm., at least, from the outline. Sulcus frontalis inferior less distinct than on the right side, evidently somewhat nearer to the frontal outline, at nearest 18 mm. distant in this view. Sulcus lunatus 20 mm. from the occipital pole of the hemisphere.

Fig. 9. — Endocranial cast. Right side.

Most conspicuous the $\hookleftarrow$ shaped sulcus frontalis inferior prolonged in a fronto-marginal notch, and the encompassed ramus anterior fissurae sylvii. Strong system of sulcus frontalis medius and sulcus frontalis superior near to the frontal outline (2 to 3 mm. distant). Clear sulcus intraparietalis (8 mm. from the outline) and sulcus lunatus nearly vertical, above the internal asterion. The latter 2 mm. below the upper border of the wide sinus transversus and 16 mm. before the posterior limit of the sinus in this view. Fissura petro-squamosa clearly indicated.

PLATE VII.

Fig. 10. — Endocranial cast. Front view.

Development of the fronto-marginal part of the sulcus frontalis medius (rectus of the Apes) very conspicuous. Clear and simple form of the right sulcus frontalis inferior; less clear and simple on the left side.

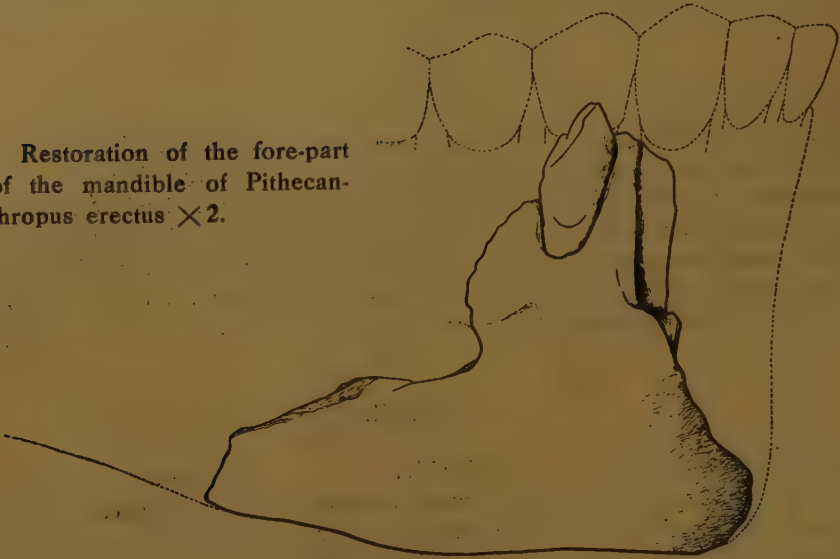
Fig. 11. — Endocranial cast. Occipital view.

The sagittal venous sinus is entirely continuous (torcular Herophili by the side of the impression of the internal occipital protuberance) into the right sinus transversus. The narrower left sinus transversus is situated much lower. On the right side very clearly the sulcus lunatus. On that side, nearly 1 cm. of the cerebellum below the transverse venous sinus still present.

PLATE VIII.

Portion of the right side of the corpus mandibulae approximate of the symphysis 2 X.

Restoration of the fore-part
of the mandible of *Pithecan-
thropus erectus* X 2.



The small mesio-distal length of the five frontmost lower teeth, in relation to the length of the three lower molars, as calculated from the length of the upper molars, at the utmost little surpassing macrodont human dimensions, is distinctly human. Human is also the ascending base-line of the lower jaw.

Fig. 12. — Outer view.

The deepest part of the notch on the left side is the fore- and lower edge of the mental foramen, placed below the posterior premolar. The summit of the fragmentary root of the anterior premolar is a little above the level of the lower border of the lost crown. Before this root the distal (posterior) surface of the somewhat flat alveolus and the root point of the canine tooth. Below the mental foramen the base-line of the mandible is somewhat prominent.

Fig. 13. — View from below.

Showing the angular lower border of the mandible, caused by the attachment of a powerful digastric muscle. The inward bending of this edge at the forepart (not clearly shown in the figure) indicates the proximity of the median line.

Fig. 14. — View from above.

The section of the root of the anterior premolar, with the canal intersected a little above its bipartition, and of the rootpoint of the canine tooth, also the mesial border of the posterior premolar socket clearly shown. Note the prominence of the outer surface of the mandible at the canine alveolus.

Fig. 15. — Mesial fracture view.

Beneath the apical part of the anterior premolar root the distal side of the canine alveolus and the canine rootpoint. The section of this anterior part of the mandibular body is still better shown on a cast, it being possible



Fig. 14.

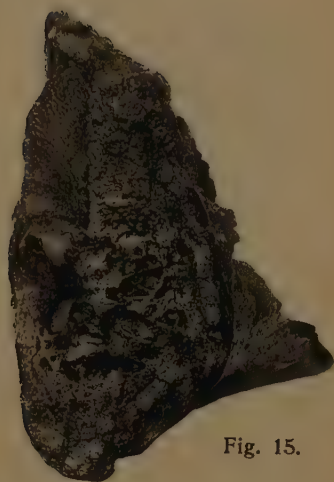


Fig. 15.



Fig. 12.

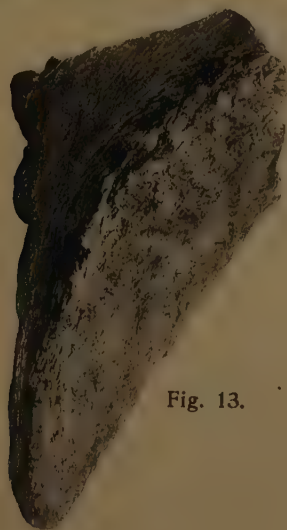


Fig. 13.



Fig. 17.



Fig. 18.



Fig. 16.



Fig. 19.



Fig. 20.



Fig. 21.



Fig. 22.



Fig. 23.



Fig. 24.



Fig. 25.



Fig. 26.

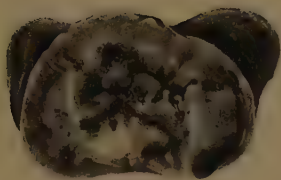


Fig. 27.

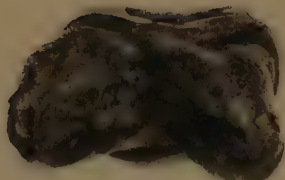


Fig. 28.



Fig. 29.



Fig. 30.

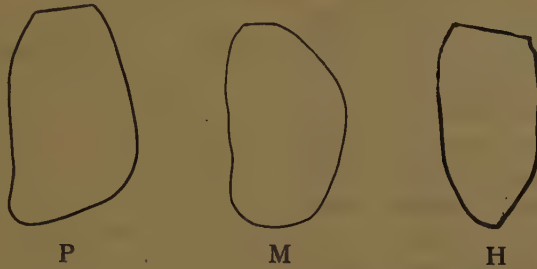


Fig. 31.



Fig. 32.

then to make the section between the canine and the anterior premolar, perpendicular to the alveolar line.



Transversal sections of casts of the corpus mandibulae (right side) of *Pithecanthropus erectus* (P) and the man of Mauer — *Homo heidelbergensis* — (M), in both mandibles, natural size, passing between the canine and the anterior premolar tooth, and section of the corpus mandibulae (right side) of a *Hylobates* (*Symphalangus*) *syndactylus* (H), passing through the socket of the first molar tooth, this section enlarged to the height of P.

The outline P is very different from M, evidently on account of the large fossa digastrica of the former mandible. The section H passes through the deepest part of the fossa digastrica. Judging from the existing part of the lower border in the fossil fragment, a section of the mandible of *Pithecanthropus* at the corresponding place resembled *Hylobates*.

PLATE IX.

Anterior left lower premolar $\times 2$.

Fig. 16. — Crown or surface view.

Human dimensions and general pattern. On the buccal surface a circle segment-like facet of wear by the upper caninus. On the mesial side an irregular concave facet of contact with the lower caninus. At the summit a strongly developed buccal cusp. No lingual cusp, on its place only confluence of the lingual and connecting transverse rim. The latter dividing the crown surface into a small anterior and a large posterior fossa. A ridge starting from the inner side of the (buccal) cusp descends in the posterior fossa. The same is seen in many anterior lower premolar crowns of anthropoid apes. Distally of (behind) the (buccal) cusp a surface of wear by the anterior upper premolar.

Fig. 17. — Mesial (anterior) view.

Showing the inward bending high buccal, the inward sloping upper and the low lingual side of the crown. Below and before the cusp the irregular facet of contact with the lower caninus. Bipartite lower part of the root, the point of the lingual-distal part broken off.

Fig. 18. — Distal (posterior) view.

Near to lingual side a crescent-like facet of contact with the lower posterior premolar.

Fig. 19. — Lingual (internal) view.

Low lingual crown side. The two fossae. Broadening of the crown upwards. Oblique position of the root (directed backwards).

Fig. 20. — Buccal (outer) view.

Circle segment-like facet of wear by the upper canine. High buccal crownside.

PLATE X.

Penultimate left upper molar $\times 2$.

Fig. 21. — Crown or surface view.

Surface smoothly worn off. The buccal-distal cusp part small, very much as in many homonymous orang-utan molars. Strongly divergent roots.

Fig. 22. — Root view.

Single lingual root. Buccal root (in this individual) composed of three fused elements, one distal (posterior) and two mesial (anterior) ones.

Fig. 23. — Mesial (anterior) view.

A large semi-ovoid facet of contact with m^1 . Buccal and lingual roots strongly divergent, the lingual root departing mostly from the vertical line.

Fig. 24. — Distal (posterior) view.

Ellipse-like facet of contact with m^3 .

Fig. 25. — Lingual (internal) view.

Direction of the roots backwards.

Fig. 26. — Buccal (external) view.

The root showing the two fused buccal elements.

PLATE XI.

Last right upper molar $\times 2$.

Fig. 27. — Crown or surface view.

Little worn. Moderately wrinkled, much less so than in Orang-Utan. Semi-ovoid as in some orang-utans, by moderate development of the distal-lingual and excessive reduction of the distal-buccal cusp. The crown as a whole shows reduction. The constriction towards the masticatory surface indicates late piercing of this tooth.

Fig. 28. — Root view. Of the strongly divergent roots the buccal one of this individual composed of three fused elements, one of which distal and two mesial.

Fig. 29. — Mesial (anterior) view.

Likewise as in Fig. 30 the lingual root mostly departing from the vertical line. No facet of contact with m^3 , probably a consequence of the still little use of the tooth on this side.

Fig. 30. — Distal (posterior) view.

Fig. 31. — Lingual (internal) view.

Fig. 32. — Buccal (external) view.

} Roots strongly directed backwards, more so than those of m^3 .

Geology. — “*Alkali Granite and Nepheline Syenites, Canadite and Foyaite, in the Vredefort Mountainland, South-Africa*” ¹⁾. By Prof. G. A. F. MOLENGRAAFF and A. L. HALL.

(Communicated at the meeting of June 28, 1924).

INTRODUCTION.

At the boundary between the Transvaal and the Orange Free State an approximately circular granite area (fig. 1) occurs, which

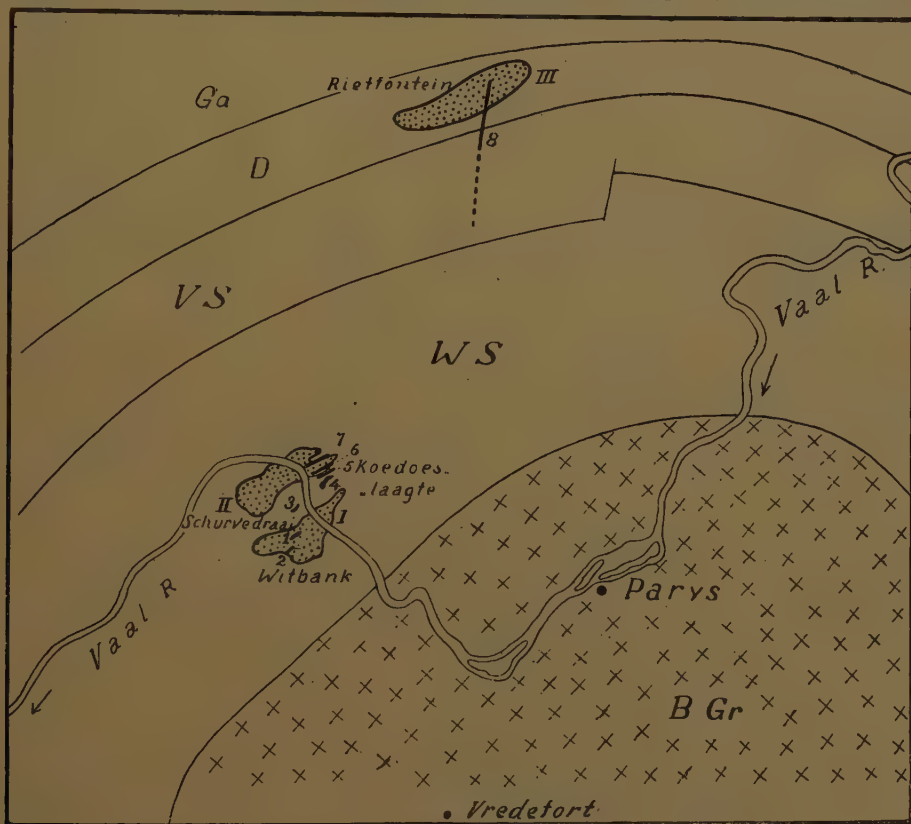


Fig. 1.

Geological sketch of a portion of the Vredefort Mountainland.

Scale $\pm 1:250,000$.

Ga = Gatsrand or Pretoria Series } Transvaal System.

D = Dolomite Series }

V. S. = Ventersdorp System.

W. S. = Witwatersrand System.

B. Gr. = Biotite Granite of Vredefort.

I, II and III = Bosses of Alkali Granite.

1—8 = Dykes of Nepheline Syenite.

The Black Reef Series, the lowermost division of the Transvaal System, between the Ventersdorp System and the dolomite, is very poorly developed in this area and has been omitted in the figures 1 and 2.

¹⁾ Shaler Memorial Series.

is intersected by the Vaal River. In this area the villages of Vredefort and Parys are situated; hence this granite is often called the granite of Vredefort or of Parys. The diameter of the area amounts to 41 km. The granite is surrounded on all sides by a girdle of sharp-crested hills composed of highly tilted or even overtilted sediments. Both the granite and the belt of sediments, which form together the Vredefort Mountainland are partly hidden from view by later deposits of Karroo age. This belt has a thickness of about 15000 m. and comprises from the granite in ascending order of succession the Witwatersrand System, the Ventersdorp System and the Transvaal System (Fig. 1 and 2). The uppermost portion of the last system

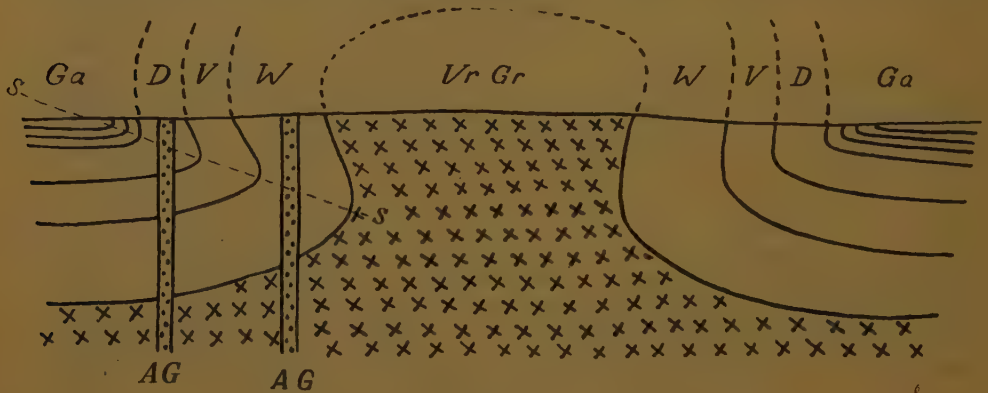


Fig. 2.

Schematic section showing the geological position of the intrusions of alkali-granite in the girdle of tilted sediments around the Vredefort granite boss.

- Ga = Gatsrand Series
- D = Dolomite Series
- V = Ventersdorp System.
- W = Witwatersrand System.
- Vr. Gr. = Vredefort Granite.
- A. G. = Alkali granite.
- S. S. = Hinge plane.

only occurs outside of the zone affected by the movements which caused the tilting of all the underlying beds. The exact age of this thick complex of very old rocks is not known; its age is at least pre-Devonian and may be pre-Cambrian.

Within this girdle of highly tilted sediments three comparatively small bosses of granite are intruded (Fig. 1). The largest occupies an area of about 5 square km. Two of these bosses are situated at the Vaal River in the Witwatersrand System on the farms Witbank, Koedoeslaagte and Schurvedraai; the third occurs in the dolomite,

the middle division of the Transvaal System, on the two farms Rietfontein.

The authors, one of whom was a member of the Shaler Memorial Expedition organised by Professor DAILY of Harvard University, discovered in the year 1922 that these three bosses consist of alkali-granite, and that dykes of nepheline-syenite occur in or near these bosses, which are intimately connected with the alkali-granites.

ALKALI GRANITE.

The alkali-granites in the three bosses are composed of medium- to fine-grained rocks and they show a great uniformity and marked petrographical family likeness. In specimens there is just enough difference to enable one to distinguish the rocks of any one boss from those of the others with certainty. In contradistinction to the much older biotite-granite or Vredefort-granite the alkali-granites nowhere show a streaky or gneissic structure, nor rapid alteration in grain.

The alkali-granites are light-coloured rather leucocratic acid rocks containing a fair amount of visible quartz, and this general character is reflected in the results of three analyses carried out by H. G. WEAIL, Government Laboratories, Johannesburg, which are given in the columns I, II and III of the table I on page 472.

During its intrusion the magma of these bosses made room for itself, partly by pushing aside the strata of the tilted sediments, partly by incorporating and assimilating material from the sediments.

The first phenomenon, the pushing aside of the pre-existing strata during their intrusion, can be best studied at the two bosses on the Vaal River (I and II in fig. 1) and is well marked by the disturbed position and the bending of the line of strike of the bars of hard quartzite near the bosses. A number of faults are combined with these disturbances and on Witbank, as well as on Koedoeslaagte and Schurvedraai, the intrusion of the two bosses of alkali-granite has given rise to very complicated tectonical relations, which have more or less affected the entire lower division of the Witwatersrand System.

The second phenomenon, the incorporation and assimilation of material from the sediments, is excellently illustrated at the first boss on Witbank and Koedoeslaagte, where a range of hills, consisting of gritty quartzites of the so-called Government Reef Series has been invaded and locally completely destroyed by the uptruding magma. In fact on both sides of the river the ridge, in which the strata of quartzite stand nearly vertical, is clean cut off against the

boss of alkali-granite. Its continuation can be followed for a small distance into the granite in the form of a rapidly fading trail of blocks of quartzite which appear to float in it. The selvedge of the alkali-granite against this quartzite is composed of a very acid hybrid rock full of xenoliths of quartzite. The first boss of alkali-granite appears to have been acidified throughout by the assimilation of all this quartzitic material, as is shown by the analysis made by H. G. WEALL, Government Laboratories, Johannesburg of an average sample of fresh rock from the centre of the boss, given in the first column of table I on page 472. The figure for silica is not less than 7%, higher than in the rock of Koedoeslaagte which in all other respects is closely allied. The analysis of the latter is given in the second column of the same table.

On Rietfontein the intrusion of the alkali-granite of the third boss has probably also been accompanied both by pushing aside of the strata of dolomite and intercalated chert and by assimilation of material from these rocks. The unsatisfactory exposure of this boss, however, prohibited to study the mode of emplacement of this granite well.

Contact-metamorphism is well developed around the bosses of alkali-granite. This phenomenon, however, is not much in evidence around those bosses which are intruded in the strata of the Witwatersrand System, because these had before the intrusion of the alkali-granites already been affected by strong thermal metamorphism; in fact in these strata the pre-existing effects of metamorphism have only been intensified at the contact with the intrusive alkali-granite.

At Rietfontein, where the boss of alkali-granite is intruded in highly tilted strata, composed of alternating layers of dolomite and chert, which had not yet suffered from any previous alteration, the contact-metamorphism is intense. In places the dolomite is altered into a beautiful, medium-grained white marble, in other places where the metamorphism is stronger it is completely converted into a very tough kind of rock, composed chiefly of a felty aggregate of needles of tremolite.

In handspecimens the alkali-granites of the Vredefort area are less trachytic than is the case with the majority of such rocks elsewhere, especially with those of Scandinavia. It is only in the alkali-granite on the two farms Rietfontein that the felspar crystals by their shape and arrangement more or less determine the general habit of the rock. The alkali-granites of the two bosses on the Vaal River could macroscopically be taken very well for ordinary biotite- and amphibole-granites.

Mineralogical composition.

The two bosses on the Vaal River consist of arfvedsonite-soda-granite. The rock which composes the bulk of the boss on Witbank (I on fig. 1) may be described here in some detail.

The leucocratic rock is chiefly composed of felspar and quartz. Microcline is the predominating felspar, generally showing lamellae after the two twin laws; where these are wanting or not seen, it is scarcely possible to distinguish the microcline from orthoclase, which may also be present in moderate proportions. Next to microcline albite is much in evidence; it forms both larger crystals with many twin lamellae and also aggregates of smaller crystals which as a rule only show a small number of twin lamellae or are simple twins. Parallel intergrowths of microcline and albite of different types are frequent; in the most common type the albite is found as a microperthitic intergrowth within the microcline; less often the latter forms microperthitic intergrowths in albite. Not seldom the albite is found in well-defined individuals in larger crystals of microcline; sometimes again this mineral is found in patches intergrown in larger crystals of albite. In several of the slides crystals of microcline or of microcline-microperthite are surrounded by a rim of albite and sometimes sinuous offshoots of the latter can be seen penetrating from the rim into the kernels of microcline. In other cases such kernels of microcline or microcline-microperthite are more or less cut up by a system of nearly parallel narrow streaks of albite. Quartz is present both in larger grains filling the interspaces left between the crystals of the other minerals, and as smaller grains forming aggregates together with small crystals of albite and microcline.

The femic minerals are all clustered together and thus appear macroscopically as dark specks fairly wide apart. Each cluster is composed mainly of arfvedsonite and aegyrine together with ore in small grains, and biotite in much varying quantities. Titanite occurs in all and zircon in the majority of the slides, confined to these clusters; apatite is present only here and there as long slender prisms.

Arfvedsonite takes the dominant place among the femic minerals. It occurs in the form of elongated prisms which show distinct faces of the prism (110), whereas in the direction of the vertical axis they are not well terminated. At both ends of this axis the crystals often run out into thin needles. Separate small sheaf-like clusters of needles and also separate needles of arfvedsonite are found scattered in small quantities, and are not seldom enclosed in the felspars.

The arfvedsonite shows the familiar strong dispersion of the

bisectrices very well and is characterized by its pleochroism:

- a deep blue
- b deep grayish blue
- c gray

Examined in ordinary light the different sections offer an exquisite variety of tints, bright lavender blue being most conspicuous under such conditions.

Among the femic minerals aegyrine takes the second place; it occurs in grains of about the same size as those of the arfvedsonite, rarely showing well defined crystal outlines. Seen in ordinary light the aegyrine in certain sections is very conspicuous by its bright emerald-green colour. In polarised light the strong pleochroism is as follows:

- a yellow to greenish yellow
- b bright green
- c deep green.

Arfvedsonite and aegyrine are often intergrown in the ordinary way, with the vertical axes and the orthopinacoidal planes parallel. These intergrowths are not seldom very intricate, a veritable mosaic being visible especially in sections parallel to the plane of symmetry.

Biotite is always present but as a rule only in small quantities. The crystals show very strong pleochroism: for rays swinging parallel to the cleavage dark olive-green to opaque, and for rays swinging at right angles to the cleavage golden-yellow.

Specks of ore are rarely wanting in any of the clusters, and also titanite in quite irregularly shaped grains is frequently found wedged in between the other femic minerals.

Plump short prisms of zircon bounded by the faces (111), (100), (010) and (110) are present in varying amounts. The crystals show a most perfect zonal structure.

The effects of the assimilation of great quantities of quartz are well seen in the mineralogical composition of the somewhat finer-grained acidified varieties of alkali-granite found in the selvedge of this boss at and near the contact with the lower quartzites of the Government Reef Series (compare page 468). There is a great increase in the amount of quartz in this selvedge rock, and numerous more or less corroded fragments of quartzite are incorporated in it as xenoliths. The feldspars do not differ from those of other parts of the rock. Aegyrine and arfvedsonite are very scantily represented, whereas biotite is present in larger quantities

than in samples taken nearer the centre of the boss. In places the rock of the selvedge carries epidote in considerable quantities.

The alkali-granite of the second boss (II in fig. 1) on Schurvedraai and Koedoeslaagte differs in minor points only from that of the first boss. The main difference is that besides arfvedsonite also another amphibole of the riebeckite-type occurs, whereas aegyrine is little represented. On Schurvedraai in the north-western portion of the boss, however, the granite differs more from that of the first boss. Biotite is there the predominant femic mineral. Arfvedsonite though present in fairly large quantities is always intergrown with and included in the larger flakes of biotite. Consequently in handspecimens biotite appears to be well-nigh the only femic mineral. Calcite occurs in the majority of the clusters of femic minerals of early consolidation in such a way that it more suggests a primary constituent than a product of decomposition.

The microscopic character of the alkali-granite of the third boss on Rietfontein (III in fig. 1) reveals certain features distinct from those of the rocks making up the other bosses. It is a soda-granite. The feldspars form the bulk of the rock. Most of their crystals are somewhat tabular parallel to (010) and are twinned after the Carlsbad law and these twins by their arrangement determine the character of the rock, by which it is individualized from the alkali-granites of the other bosses. Each of the components of the Carlsbad twins consist of parallel intergrowths of orthoclase, microcline, microcline-microperthite and albite, which show a great variability. The rims of the crystals generally consist of albite. The components of the Carlsbad twins as a rule meet along an irregular composition plane.

Besides feldspar, quartz, amphibole, biotite, zircon and ore are always present, and pyroxene and titanite are observed here and there.

The femic minerals are clustered together. The amphibole is idiomorphic and well crystallized, the main crystalform being a somewhat stout prism. Twins after the orthopinacoid are of common occurrence. The pleochroism is strong:

- a light yellowish green
- b deep olive-green
- c deep olive-green with a shade of blue.

The amphibole belongs to a type between common hornblende and arfvedsonite. Pyroxene is wanting in many of the slides and in others appears to represent a type between aegyrine and diopside.

The biotite, which occurs in good-sized flakes, is a strongly pleochroic lepidomelane. The colours shown for rays swinging at

right angles to the cleavage are golden yellow to brownish yellow, and for these swinging parallel to the cleavage, deep brown to opaque; smaller flakes of biotite are frequently enclosed in the amphibole and these form not seldom parallel intergrowths with their host. Ore, probably magnetite, as black specks is found scattered in the rock and also enclosed both in the crystals of biotite and of amphibole. Zircon here and there joins the clusters of femic minerals and so does sphene. Quartz which is much in evidence fills the interspaces between the other minerals of the rock.

Chemical composition.

The chemical composition of the alkali-granites of the Vredefort Mountainland and also of some allied rocks, is shown in the following table.

TABLE 1.

	I	II	III	IV	V	VI	VII
SiO ₂	74.35	67.3	70.65	70.59	71.24	71.65	70.40
TiO ₂	0.15	0.5	0.3	0.44	0.68	trace	0.13
ZrO ₂		0.10	0.05				1.65
Al ₂ O ₃	13.35	16.65	15.25	12.38	13.78	13.04	7.85
Fe ₂ O ₃	1.35	2.5	1.7	1.61	1.30	2.79	6.85
FeO	1.1	1.75	1.65	3.33	2.83	1.80	2.98
MnO	0.05	trace	trace	0.08	0.15		0.13
CaO	0.85	0.95	0.85	0.93	0.38	trace	0.26
MgO	0.45	0.75	0.5	nil	trace	trace	0.52
Na ₂ O	5.1	6.2	5.2	6.95	5.32	6.30	4.05
K ₂ O	2.9	2.6	3.5	3.74	5.10	3.98	4.25
P ₂ O ₅	0.1	0.15	nil	trace			
CO ₂	nil	nil	nil				
H ₂ O at 110°	0.15	0.3	0.2	0.21		1.10	
H ₂ O, loss on ignition	0.3	0.4	0.35	0.20			0.25
	100.20	100.15	100.20	100.46	100.78	100.66	99.65

I. Arfvedsonite-soda-granite. Centre of boss 1, Witbank. Analyst H. G. WEALL, Government Laboratories, Johannesburg. Probably a hybrid, acidified rock.

II. Arfvedsonite-soda-granite, boss 2, Koedoeslaagte. Analyst H. G. WEALL, Government Laboratories, Johannesburg.

III. Soda-granite, boss on Rietfontein. Analyst H. G. WEALL, Government Laboratories, Johannesburg.

IV. Arfvedsonite-soda-granite, Ilimausak, South Greenland. Analyst C. WINTHER.

V. Soda-granite, Iviangusat, Kangerdluarsuk. South-Greenland. Analyst C. DETLEFSEN. A hybrid acidified rock.

VI. Arfvedsonite-soda-granite, Hougatten, Loughnatten, Norway. Analyst L. SCHMELCK.

VII. Arfvedsonite-granite, Ampasibitika, Madagascar. Analyst M. F. PISANI.

The analysed rocks of the Vredefort area all represent types in which amphibole is the dominant femic mineral. They are soda-granites, soda occurring much in excess over potash. Iron is present in small quantities only, and magnesia and lime in very small quantities.

Such leucocratic soda-granites are uncommon rocks. They have a great similarity to the soda-granites described by BRÖGGER¹⁾ which occur intimately connected with the Nordmarkites in the Christiania area. They are further closely related to the arfvedsonite-granite of Ilimausak (IV in table) and the soda-granite of Iviangusat, (V in table) both described by USSING²⁾ and the soda-granite of Hougatten (VI in table), described by BRÖGGER³⁾. The arfvedsonite-granite of Ampasibitika in Madagascar (VII in table) described by LACROIX⁴⁾ is much richer in iron than the soda-granites of the Vredefort-area.

The abundance of microcline among the feldspars approaches these rocks also to the riebeckite-soda-granite of San Peter's Dome, Colorado, described by LACROIX⁵⁾.

NEPHELINE SYENITE.

Dykes of nepheline-syenite occur in intimate connection with the bosses of alkali-granite described in the previous pages.

¹⁾ W. C. BRÖGGER, Die Mineralien der Syenitpegmatitgänge der Südnorwegischen Augit- und Nephelingsgesteine. Zeitschr. für Kryst. XVI, pp. 65—70, 1890.

²⁾ N. V. USSING, Geology of the country around Julianahab, Greenland. Meddelelser om Grønland. Vol. XXXVIII, p. 114, Copenhagen 1911.

³⁾ W. C. BRÖGGER, Die Eruptivgesteine des Kristianiagebietes, I, p. 127, Kristiania 1894.

⁴⁾ A. LACROIX, Matériaux pour la minéralogie de Madagascar. Nouvelles Archives du Muséum (4) I, p. 82, Paris 1902.

⁵⁾ A. LACROIX, Sur une roche à amphibole sodique (riebeckite), astrophyllite, pyrochlore et zircon du Colorado. Comptes Rendus, Tome 109, p. 39, 1889.

The dykes of nepheline-syenite either cut through the bosses of alkali-granite, whence they may extend into the surrounding sedimentary strata, as is the case at the second boss on Koedoeslaagte and the boss on Rietfontein, or they may occur in the immediate vicinity of those bosses, as is the case with the dyke between the bosses I and II both on Witbank (Locality 3) and on Koedoeslaagte (Locality 4). The dykes have a varying width, in the case of the Canadites not exceeding 16 m. (Dyke Locality 7 on Koedoeslaagte). They stand vertical, as far as this could be proved with certainty, and their strike corresponds on the average (not on Rietfontein) with the general strike of the invaded sediments. The dykes of nepheline-syenite though genetically connected with the bosses of alkali-granite are younger than those and must have been injected after the emplacement of the masses of alkali-granite; consequently they are also younger than the phenomenon of up-doming of the Vredefort granite boss. They are, just as well as the alkali-granites, much altered by pressure and crushed and consequently in many places converted into mylonites and ultimately in pseudo-tachylites or flinty crush-rocks. Numerous veins composed of these latter rocks cut through the alkali rocks in various directions; on Plate I in fig. 1 such a vein is shown occurring is a canadite of Locality 5 (see fig. 1).

Just as in the alkali-granites two groups could be distinguished, on the one hand the typical arfvedsonite-soda-granites of the bosses I and II and on the other hand the soda-granite of boss III, so the nepheline-syenites connected with the first two bosses are as a group distinct from those which are found connected with the third boss.

The dykes of the first group on Witbank, Koedoeslaagte and Schurvedraai notwithstanding considerable variety in external appearance all consist of Canadites, whereas the dyke on Rietfontein must be classed amongst the Foyaïtes.

THE CANADITES AT THE VAAL RIVER.

Although the nepheline-syenites of the different dykes on Witbank, Schurvedraai and Koedoeslaagte show in many places rapid variations of grain, structure and mineralogical composition yet they all have a strong family likeness. They belong to the group of the Canadites, defined by QUENSEL and SHAND as nepheline-syenites with nepheline and albite as principal salic minerals. QUENSEL¹⁾ first proposed the

¹⁾ P. QUENSEL, The alkaline rocks of Almunge, Bull. Geol. Inst. of Upsala XII, p. 177, Upsala 1914.

name Canadite for such rocks, but in defining the group he emphasizes the abundance of femic minerals, whereas SHAND calls the canadites „rather leucocratic rocks”. In this respect the rocks of the Witbank area best correspond to the type as it is defined by SHAND¹⁾. for on the average they are decidedly leucocratic rocks.

The general characteristics given by ADAMS and BARLOW²⁾ of the nepheline-syenites of the Haliburton and Bancroft areas on pp. 228—232 of their memoir can to a great extent be applied to the majority of the nepheline-syenites of the Witbank area. The canadites of this area, however, do not contain calcite and in places are free from aegyrine and rich in lepidomelane, when they correspond to the litchfieldite-type, as described by BAILEY³⁾.

The mineralogical composition is varying, but all the varieties observed pass one into the other, generally rather abruptly. Although each of the dykes shows one or more characteristics of its own, by which it can be singled out, such characteristics mostly apply to the way in which the different varieties of the canadite-type are represented.

The rocks are usually massive, but in some places they possess a more or less perfect banded or even schistose structure reminding of that of lujaurites, (Pl. I, fig. 2). Massive and banded streaks may alternate repeatedly. They vary in texture from fine- to coarse-grained or even porphyritic (Pl. I fig. 1) while in pegmatitic phases nepheline and anorthoclase occur as individuals as much as 20 cm. in diameter.

The coarse-grained and porphyritic phases may occur as more or less parallel streaks intercalated in finer-grained rock, especially where the latter is more or less banded (localities 4 and 7), in other places the pegmatite may occur in irregular patches (localities 1, 4 and 7) and also the rock may present quite a massive development over a good distance, which at once is interrupted by the appearance of clusters or phenocrysts of large dimensions (locality 3).

The rocks, as a rule, are quite fresh and unaltered and only show a thin weathered coating.

Mineralogical composition.

The canadites of Witbank and Koedoeslaagte are made up essen-

¹⁾ S. J. SHAND, The nepheline rocks of Sekukuniland, Trans. Geol. Soc. of S.-Africa, XXIV, p. 117 and 118, 1921.

²⁾ F. D. ADAMS en A. E. BARLOW, Geology of the Haliburton and Bancroft areas, Geol. Survey Canada, Memoir Nr. 6, Ottawa 1910.

³⁾ W. S. BAYLEY, Eleolite-syenite of Litchfield, Maine, Bull. Geol. Soc. of America III, pp. 231—251, 1892.

tially of albite and closely allied plagioclases, anorthoclase, nepheline, cancrinite, aegyrine and biotite; also microcline, orthoclase, amphibole, titanite, apatite and magnetite enter into the composition of the rocks.

The order of crystallization is not quite definite. Magnetite, apatite and sphene were the first minerals to crystallize, and after those aegyrine and biotite were formed; albite and allied feldspars came next in order; nepheline as a rule is crystallized somewhat later than the feldspars and cancrinite is the last mineral to crystallize.

Numerous exceptions, however, have been observed to this general order of crystallization; thus nepheline may be found included in feldspar, and lepidomelane may poikilitically embrace such salic minerals as albite and nepheline.

Among the minerals of which the canadites are composed rank as main constituents feldspars, nepheline, cancrinite, aegyrine, biotite; and as accessory constituents amphibole, muscovite, titanite, apatite and magnetite.

Main constituents. Feldspars. The feldspars as a rule form the bulk of the rocks, although locally nepheline may preponderate over feldspar. The crystals are fresh and transparent even at a small distance from the weathered surface.

Albite, with allied acid plagioclases, is by far the ruling feldspar in the massive portions of the rocks. The crystals of albite are more or less tabular after the brachypinacoid; the smaller crystals are as a rule simply twinned, the larger ones show repeated twinning and then the lamellae are often so close together, that they become difficult to distinguish. The anorthoclase occurs in rather large more irregular plates with either no albite lamellation or only a very shadowy indication, or with a patchy form of extinction resembling quartz with undulatory extinction. Anorthoclase is the dominant if not the only feldspar amongst the phenocrysts. Its crystals often attain large dimensions, and diameters of about 15 cm. are not rare. The dyke 3 (see Fig. 1) carries such phenocrysts in abundance. Both in the massive portions and in the groundmass of the porphyritic portions of the rock anorthoclase plays an unimportant role compared with albite. In its somewhat tabular crystals the albite lamellation is only faintly indicated. Microperthite is always present intergrown with albite but never as abundant as in the soda-granites. Microcline is erratic, wanting in many of the rocks and rather abundant in some of them. Orthoclase is rarely observed with certainty and is doubtless a subordinate mineral in these rocks.

Nepheline. The nepheline is quite fresh; under the microscope it is colourless and transparent. Macroscopically on fresh fractures of

the rock it is recognisable by its greasy lustre; its colour is gray to faint greenish. In the porphyritic varieties nepheline occurs both in the groundmass and as phenocrysts (Pl. I, fig. 1). The latter have the shape of short plump prisms, bounded by the faces of the basal pinacoid and the prism (1010); they may attain a diameter of about 10 cm. as e.g. in the dyke at locality 4.

On the weathered surface of the rock the crystals of nepheline by their relatively rapid weathering are invariably indicated and easily discernable by pits or depressions surrounded by rims composed of minerals such as feldspar or aegyrine. The shape of these pits or depressions is either hexagonal or quadratic. The surface of the nepheline in these depressions is often shagreened and coated by a thin film of a bluish gray enamel¹).

The crystals of nepheline in the groundmass may be idiomorphic and then they often show a perfect zonal arrangement of their inclusions which as a rule chiefly consist of needles of aegyrine. The figure 4 in Pl. I represents a section of a crystal of nepheline cut at right angles to the optical axis showing this zonal arrangement. In other rocks the crystals of nepheline are hypidiomorphic and are bounded by crystal faces in the zone of the prism only, but bluntly terminated in the direction of the optical axis. As a rule such crystals have fairly large dimensions and the enclosed needles of aegyrine are then arranged exclusively parallel to the optical axis. In other rocks again the nepheline is perfectly allotriomorphic and is found wedged in between the crystals of feldspar in a similar way as quartz in granites. Besides aegyrine all the other minerals, cancrinite excepted, may be found enclosed in nepheline, but never in abundance.

Cancrinite. This mineral although present in notable quantities in several nepheline-syenites of the Witbank area can only be distinguished by the aid of the microscope. It occurs in irregular perfectly allotriomorphic grains of about the same size as those of albite and nepheline in the non-porphyritic portions of the rocks. It also occurs as thin beady rims surrounding crystals of feldspar or nepheline, and again it may enter into cracks or fissures, especially in feldspars. This may explain, why it is sometimes found apparently enclosed in feldspar.

Under the microscope the cancrinite is transparent and colourless, free from inclusions, quite fresh and consequently free from alteration products. It always occurs somewhat concentrated in streaks and

¹) A similar coating is described by ADAMS and BARLOW, l.c. p. 236.

patches and in some of the dykes streaks are found, where the cancrinite occurs in quantities great enough to make it one of the dominant minerals. Where this occurs the name cancrinite-canadite would be appropriate. An exceptionally fine variety of this type is found in the banded portion of the big dyke (locality 7 in fig. 1) on Koedoeslaagte, which is a granular rock made of grains of about equal dimensions of albite, nepheline and cancrinite without femic minerals (Pl. I, fig. 3). Albite is the preponderating mineral, next comes cancrinite and nepheline takes the third place. Nepheline is scarce or even wanting in some portions of this rock. This cancrinite-canadite sometimes contains as the sole femic constituent lepidomelane in such quantities that the name cancrinite-litchfieldite would be justified.

The cancrinite is a primary constituent, and even where it occurs in very small grains, arranged as beady rims bordering such minerals as nepheline and felspar it does not make the impression of a secondary element.

Aegyrine and biotite. These minerals are never met with in equal quantities, either one or the other dominating. In much of the occurrences pyroxene is the ruling femic constituent. The pyroxene as to its optical characteristics conforms to aegyrine but the pleochroism is much less strong than is usually observed in aegyrine. In the kernels of the crystals the pleochroism, although faint is well discernable

a bright green

b light green

c pale yellow

whereas the rims show hardly any pleochroism. The pyroxene occurs chiefly in the form of long and slender prisms which may take the shape of needles; not rarely, however, the prisms are stouter and less elongated, and sometimes they are more or less tabular after the orthopinacoid.

Biotite is an important mineral only in those varieties, which belong to the litchfieldite-type chiefly found in the dyke at locality 7. This rock carries biotite in abundance both in the porphyritic portions as flakes which may attain a diameter of 14 cm., and in smaller crystals scattered through the rock. The phenocrysts show on their cleavage planes the traces of the three systems of gliding planes well. The mineral is a lepidomelane with very strong pleochroism from straw-yellow to opaque. It not rarely poikilitically encloses crystals and grains of all the other minerals, with the exception of the cancrinite.

Accessory constituents. Hornblende. This mineral is rare and found in notable quantities only in a dyke on Koedoeslaagte (Locality 6) as small prisms together with and in between prisms of aegyrine. Its pleochroism is strong

a deep bluish green to opaque

b dark olive-green

c yellowish green.

This amphibole shows the optical properties of arfvedsonite, only the pleochroism does not conform to that of arfvedsonite.

Muscovite. This mica is exclusively found in some rocks of the litchfieldite type, which are rich in cancrinite. It is possible, but not certain that this muscovite is a product of decomposition.

Titanite. This mineral is on the average well represented amongst the accessory minerals, in microscopic crystals only. It is erratic in its occurrence, sometimes much in evidence and sometimes very scarce or absent. It is generally found in irregular grains and with aegyrine.

Apatite is rare and only observed in some of the dykes. It is well crystallized either in slender or plump prisms, only visible under the microscope.

Magnetite. This mineral is always present in specks or more frequently in well-defined microscopical crystals with triangular, quadratic or hexagonal outlines. Sometimes the crystals are scattered all through the rock but more often they are clustered together with or enclosed in flakes of lepidomelane or aegyrine.

Zircon is present in the majority of the rocks. The colour of this mineral is yellowish and it is conspicuous by the beautiful zonar structure of the crystals, which are bounded by the faces of prism and pyramid.

Chemical composition.

The chemical composition of the canadites on the Vaal River is given in the columns I and II of the following table.

I. Coarse Nepheline Syenite from Locality 7 on Koedoeslaagte, Analyst H. G. WEALL, Government Laboratories, Johannesburg.

II. Fine-grained Nepheline Syenite from Locality 7 on Koedoeslaagte. Analyst H. G. WEALL, Government Laboratories, Johannesburg.

III. Dark-coloured canadite, Byske, Almunge district, Sweden. Analyst M. DITTRICH.

TABLE 2.

	I	II	III	IV	V	VI
SiO ₂	57.3	61.65	48.60	51.58	60.39	57.78
TiO ₂	0.3	0.2	1.34	0.35	—	1.83
ZrO ₂	nil	nil	trace	—	—	—
Al ₂ O ₃	22.3	20.5	19.89	19.40	22.57	15.45
Fe ₂ O ₃	4.1	3.5	2.97	4.26	0.42	3.06
FeO	1.1	0.7	5.76	5.25	2.26	3.11
MnO	trace	trace	0.36	0.20	0.08	0.98
CaO	2.75	2.75	4.43	3.64	0.32	1.72
MgO	0.25	0.35	1.32	0.49	0.13	1.13
Na ₂ O	9.1	8.8	8.74	7.49	8.44	11.03
K ₂ O	1.95	0.85	2.26	4.23	4.77	2.89
P ₂ O ₅	nil	nil	0.56	0.15	—	—
CO ₂	nil	nil	1.10	1.53	trace	—
H ₂ O, at 110°	0.2	0.15	0.21	—	—	0.94
H ₂ O, loss on ignition	0.45	0.5	1.73	1.02	0.57	
	99.8	99.95	99.27	99.59 ¹⁾	99.95	99.92

¹⁾ Including SO₂ 0.10; F 0.06; S 0.01; BaO 0.05; Ce₂O₃ 0.59.

IV. Nepheline-syenite (canadite), Monmouth Co., Ontario, Canada. Analyst M. F. CONNOR.

V. Litchfieldite, Litchfield, Maine. Analyst L. G. EAKINS.

VI. Nepheline-syenite (arfvedsonite-canadite), Tuoljbuht, Finland. Analyst H. BERGHELL.

The great preponderance of soda over potash, combined with the fairly low percentage of calcium, explains the scarcity of orthoclase and the abundance of albite, and of plagioclase poor in anorthite. The absence of carbon-dioxide in the analyses, notwithstanding the common occurrence of cancrinite in these canadites, can only be explained by the fact that this mineral is somewhat erratic in its occurrence. The analyses are evidently made from fragments in which cancrinite is absent or nearly so.

Comparing the analyses of these rocks with those of other localities, besides the above mentioned strong preponderance of soda

over potash, the relatively high percentage of silica and the scarcity of iron also attract attention. The latter features explain the pronounced leucocratic character of the rocks.

The chemical composition has a similarity, but not a close one, to those of the canadites of Almunge and of Monmouth County which served QUENSEL¹⁾ as types in distinguishing his special group of *canadites*. The composition of these rocks is given for comparison in columns III and IV. The chemical composition of the variety of canadite containing lepidomelane as the sole femic constituent, to which BAYLEY²⁾ has given the name *litchfieldite* is added in column V.

The canadites of the Vaal River contain more silica, and less iron and calcium than the canadites of Almunge and Monmouth County. One has, however, to take into account that in the case of the Almunge-rock only the dark-coloured commonest type is analysed.

The Almunge-rock, however, shows a great diversity of types, and the dark-coloured varieties, rich in femic minerals, may grade rapidly into white or pink rocks devoid of dark minerals; these latter leucocratic varieties are not analysed. From the descriptions it appears that in the Almunge-district and in Monmouth Co. melanocratic rocks greatly preponderate, whereas in the Witbank area by far the bulk of the nepheline-syenites is decidedly leucocratic, the melanocratic varieties being restricted to a few bands and streaks only.

Thus the nepheline-syenites of the Witbank area appear to represent one of the purest leucocratic types, hitherto found, of the canadite group, i. e. of a nepheline-syenite composed chiefly of nepheline and albite. As far as can be judged from the description and the analysis the nepheline-syenite of Tuolj-bucht on the peninsula of Kola³⁾ offers another excellent example of a leucocratic canadite. This rock is poorer in nepheline than the canadites of the Witbank area and contains as femic minerals arfvedsonite and some aegyrine. Its chemical composition is given in column VI.

Amongst the nepheline-syenites found elsewhere in Transvaal, in the Bushveld, no canadites have been recorded.

THE FOYAITE ON RIETFONTEIN.

The foyaite on the farms Rietfontein N°. 555 and N°. 664 occurs

¹⁾ P. QUENSEL, l.c. p. 180.

²⁾ W. S. BAYLEY, l.c. p. 241.

³⁾ W. RAMSAY und V. HACKMAN, Das Nephelinsyenitgebiet auf der Halbinsel Kola I, Fennia II, p. 139, 1894.

as a powerful dyke ¹⁾ cutting through the boss of alkali-granite described above and the surrounding dolomite. Its strike is N8 E., its dip about 90°. The thickness is 28 metres. It can be followed for about 2 km. from near the boundary between Rietfontein 555 and 664 in a north-northeasterly direction to the central portion of the first farm along a series of outcrops well marked by accumulations of large rounded blocks.

The dyke of weathered foyaite which occurs 6 k.m. more to the South on Buffelshoek probably is the continuation of this dyke on Rietfontein.

The foyaite is a pale-pink medium-grained slightly porphyritic rock. The handspecimens show pale-pink orthoclase, often white by decomposition and many irregular waxy-grey to dark-red patches of nepheline. The dark minerals aegyrine, aegyrine-diopside and biotite are scattered freely through the rock in small irregular clusters.

Mineralogical composition.

Thin sections show the following minerals in order of abundance: orthoclase, nepheline, pyroxene, and biotite.

Orthoclase. This is the chief component; some crystals are larger than the others and appear as phenocrysts. They are all tabular after (010). The phenocrysts are perfectly euhedral and reach a length of 15 mm, parallel to the *c*-axis, whereas the thickness of the platy crystals in the direction of the *b*-axis does not exceed 1 mm. Carlsbad twins are frequent. Both the phenocrysts and the smaller crystals of orthoclase have their rims generally composed of albite, while in the kernels the orthoclase is intergrown in small quantities with microperthite and albite. The orthoclase is more or less decomposed and cloudy but the albite is quite transparent and fresh.

Nepheline. Under the microscope the nepheline is more idiomorphic than the handspecimens suggest. Although the crystals appear to be wedged in between the tabular feldspars they are often euhedral and bounded by faces of the prism and of the basal pinacoid. The bulk of the nepheline is fresh and transparent, though in part turbid with the properties of elaeolite; inclusions are few.

Pyroxene. The pyroxenes are represented by diopside and aegyrine.

¹⁾ This dyke has been discovered and described as a dyke of elaeolite-syenite by E. T. MELLOR in: The geology of the central portion of the Potchefstroom District, Report of the Geol. Survey for the year 1907, p. 26. Pretoria, 1908.

The former mineral occurs in plump prisms bounded by the faces of the two pinacoids (100) and (010) truncated by small faces of the prism (110). The aegyrine occurs in well defined tabular crystals flattened parallel to the orthopinacoid. In the vertical zone the crystals are bounded by the faces of the orthopinacoid and the prism (110). The diopside shows a maximum angle of extinction with the axis c of 42° in sections parallel to the plane of symmetry; those of aegyrine in corresponding sections have a nearly straight extinction.

In the crystals of diopside the pleochroism is weak:

- a yellowish grey
- b greenish grey
- c greenish to bluish grey.

In the crystals of aegyrine the pleochroism is strong:

- a dark bluish green
- b emerald green
- c greenish yellow.

The crystals of diopside, often have broad rims composed of aegyrine, the two minerals forming together parallel intergrowths.

Biotite is always clustered together with the other femic constituents. It is a dark brown variety showing pleochroism from orange-brown to opaque. Magnetite is always found in the rock, but apatite is very scarce, and sphene has not been observed.

The order of crystallization is not well marked in these rocks and the separation of all the component minerals appears to have continued up to the moment of final consolidation. Although the felspar is more euhedral than any of the other minerals yet in places its tabular crystals are influenced in their position by pre-existing faces of nepheline. Pyroxene is largely idiomorphic and yet felspar laths penetrate into pyroxene and biotite, and locally the spots where the femic minerals are clustered together then show an ophitic structure. Biotite is in places found with good crystal faces, but in many of the clusters it embraces poikilitically all the other constituents.

Chemical composition.

The nepheline-syenite of Rietfontein belongs to the group of the foyaites.

Its chemical composition is given in column I of the following table.

TABLE 3.

	I	II	III
SiO ₂	55.3	56.12	56.44
TiO ₂	0.8	0.46	1.16
ZrO ₂	0.15	—	—
Al ₂ O ₃	18.25	19.62	15.54
Fe ₂ O ₃	3.55	2.32	3.27
FeO	3.45	0.90	3.67
MnO	0.6	—	—
CaO	3.15	2.07	4.16
MgO	0.55	0.13	1.73
Na ₂ O	5.6	9.50	5.81
K ₂ O	4.65	4.17	4.27
P ₂ O ₅	0.15	—	0.83
CO ₂	nil	0.80	0.97
H ₂ O, at 110° C.	0.6	—	0.44
H ₂ O, loss on ignition	3.00	3.50	2.06
	99.80	100.39	100.35

I. Nepheline-syenite, foyaite, Rietfontein 555. Analyst H. G. WEALL, Government Laboratories, Johannesburg.

II. Foyaite, normal type. Leeuwfontein 320, Transvaal. Analyst M. F. PISANI.

III. Analcite-syenite, Mauchline, Ayrshire, Scotland. Analyst M. DITTRICH.

In the columns II and III the composition of two chemically analogous nepheline-syenites is given, viz. a Foyaite of Leeuwfontein described by BROUWER¹⁾ and an analcite-syenite of Mauchline described by TYRRELL²⁾.

RELATIONSHIP BETWEEN THE ALKALI-GRANITES AND THE NEPHELINE-SYENITES.

A close analogy in chemical composition exists between the alkali-

¹⁾ H. A. BROUWER, Oorsprong en samenstelling der Transvaalsche Nephelien-syenieten. 's-Gravenhage 1910, p. 68 en 91.

²⁾ G. W. TYRRELL, The late palaeozoic alkaline igneous rocks of the West of Scotland, Geol. Mag. 5, vol. IX, pp. 69—80 (1912).

granite and the nepheline-syenite of the dykes, which are connected with the bosses of alkali-granite. This analogy becomes obvious as soon as one applies the so-called triangular method of projection of OSANN¹⁾, which makes independent of the content of silica. The projection of all the analyses, made of the alkali-granites and nepheline-syenites of the Vredefort Mountainland, with exception of the foyaite of Rietfontein, lie closely together in OSANN's triangle and partly even coincide. The somewhat distant position of the foyaite of Rietfontein is caused by the fairly great amount of iron contained in that rock. It is allowed to infer from these facts, that the magma, which has been injected in the dykes of nepheline-syenite, probably has been a part of the magma of the alkali-granite which had lost a portion of its silica.

The authors intend to describe these alkaline rocks more in detail in a memoir on the Vredefort Mountainland, and then to discuss also briefly their probable mode of origin in connection with other rocks in that area. They will confine themselves now to call the attention to the fact that the soda-granite and the foyaite on Rietfontein have been intruded into rocks of the dolomite formation. DALY's²⁾ theory on the genesis of alkaline rocks would be readily acceptable here, as soon as one admits, that the acid magma rich in soda — probably a soda-rich and acid differentiate or residue of a much larger hidden magma of more basic composition — which has been intruded into the dolomite and is found now solidified in the boss of soda-granite, has absorbed in the depth a certain quantity of the dolomitic rock, causing it to become locally desilicated. From these desilicated portions then originated the dykes of foyaite.

A similar explanation would appear to be untenable in the case of the nepheline-syenites on the Vaal River, because these are injected in the Witwatersrand System, of which no limestones or dolomites form part. Yet one has to take into account, that the tilting and overtilting of the belt of sediments round the Vredefort granite-mass has been accompanied by strong tectonic disturbances and dislocations. It is not quite impossible that along the hinge plane SS (fig. 2) disturbances caused by faulting and shearing were great enough to allow masses derived from the dolomite to come

¹⁾ A. OSANN, Versuch einer chemischen Classification der Eruptivgesteine T. M. P. M. Bd. XIX, p. 351, 1900.

²⁾ R. A. DALY, Origin of the alkaline rocks, Bull. Geol. Soc. of America. Vol. XXI, p.p. 87—118, 1910. Compare also SHAND's modification of DALY's theory: S. J. SHAND, The problem of the alkaline rocks. Presid. Address Geol. Soc. of S. Africa 1922 pp. XIX—XXXII.

in contact with and to get incorporated into the ascending magma. It certainly would be difficult to explain the presence of notable quantities of cancrinite in the nepheline-syenites on the Vaal River, in case the possibility of rocks containing carbonates being absorbed in the alkaline magma were considered to be absolutely non-existent.

The authors feel greatly indebted to Mr. H. J. WEALL of the Government Laboratories at Johannesburg, who made the analyses of the rocks and to Dr. P. KRUIZINGA at Delft, who carefully prepared the microphotographs.

Delft, June 1924.

Geological Laboratory.

G. A. F. MOLENGRAAFF and A. L. HALL: "Alkali Granites and Nepheline Syenites, Canadite and Foyaite, of the Vredefort Mountainland, South-Africa."

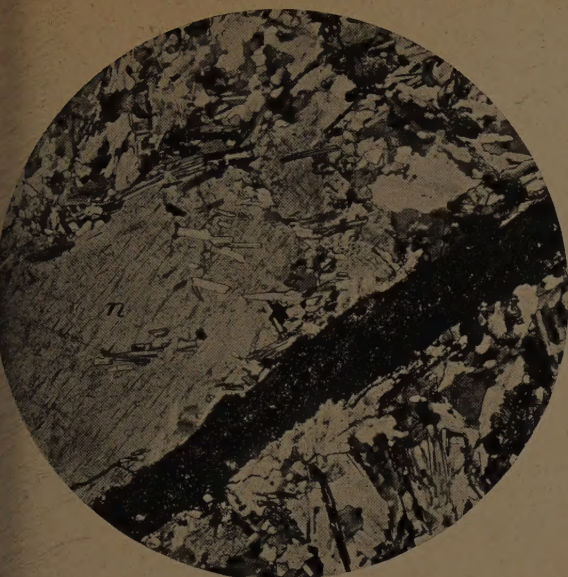


Fig. 1.

Porphyritic type of canadite, cut by a vein of pseudo-tachylite; n. nepheline, enclosing crystals of albite. Koedoeslaagte, dyke 5 in fig. 1. Magn. 25 $\times$. + nicols.



Fig. 2.

Canadite with parallel arrangement of crystals of a. albite, ae. aegyrine, n. nepheline. Koedoeslaagte, dyke 7 in fig. 1. Magn. 42 $\times$. + nicols.



Fig. 3.

Cancrinite-canadite, c. cancrinite, n. nepheline. Koedoeslaagte, dyke 7 in fig. 1. Magn. 70 $\times$. + nicols.

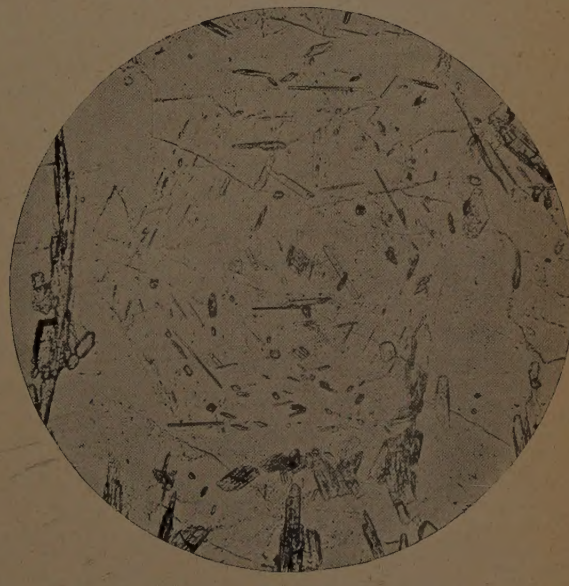


Fig. 4.

Section of a crystal of nepheline parallel to the basal pinacoid; needles of aegyrine arranged parallel to the faces of the prism. Koedoeslaagte, dyke 4 in fig. 1. Magn. 148 $\times$, ordinary light.

Mathematics. — “*Perfect sets of points with positively-irrational distances*”. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of February 23, 1924).

The following construction of a set of points of the kind mentioned in the title is perhaps somewhat simpler than that which was developed by Prof. WOLFF in these Proceedings Vol. XXVII, p. 95—96.

In the closed unity interval we determine a fundamental series of sets of points $s_1, s_2, s_3, \dots$, in which each s_v consists of 2^v closed intervals having distances greater than zero from each other, whilst within each interval of s_{v-1} two intervals of s_v are situated, and the distance of two positively-different points of s_v differs positively from each fractional number $\frac{a}{n}$ ($n \leq v$). The greatest common divisor $\mathfrak{D}(s_1, s_2, \dots)$ forms a perfect set of points two arbitrary positively-different points of which possess a positively-irrational distance.

In the above fundamental series the possibility of determining s_v when disposing of s_{v-1} follows from the property that in the closed unity interval for each positive integer v and for each positive ϵ a finite set of closed intervals can be defined approaching each point of the closed unity interval at a distance $< \epsilon$, and two arbitrary positively-different points of which have a distance positively differing from each fractional number $\frac{a}{v}$.

This definition can be given as follows: Let m be a positive integer such that the greatest common divisor of m and v be unity, whilst $\frac{1}{m} \leq \epsilon$, then the distance of two arbitrary points of the finite set of

points π_m consisting of the points $\frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}$, differs at least

an amount of $\frac{1}{mv}$ from each non-vanishing fractional number $\frac{a}{v}$.

So, if on each point of π_m as centre we lay a closed interval of length $\frac{1}{2mv}$, a finite set of closed intervals arises approaching each point of the closed unity interval at a distance $< \epsilon$, whilst two arbitrary positively-different points of it have a distance differing at least an amount of $\frac{1}{2mv}$ from each non-vanishing fractional number $\frac{a}{v}$.
